Chapter 2

Fourier Series & Fourier Transform

Updated:2/11/15

Outline

- Systems and frequency domain representation
- Fourier Series and different representation of FS
- Fourier Transform and Spectra
- Power Spectral Density and Autocorrelation Function
- Orthogonal Series Representation of Signals and Noise
- Linear Systems
- Bandlimited Signals and Noise
- Discrete Fourier Transform

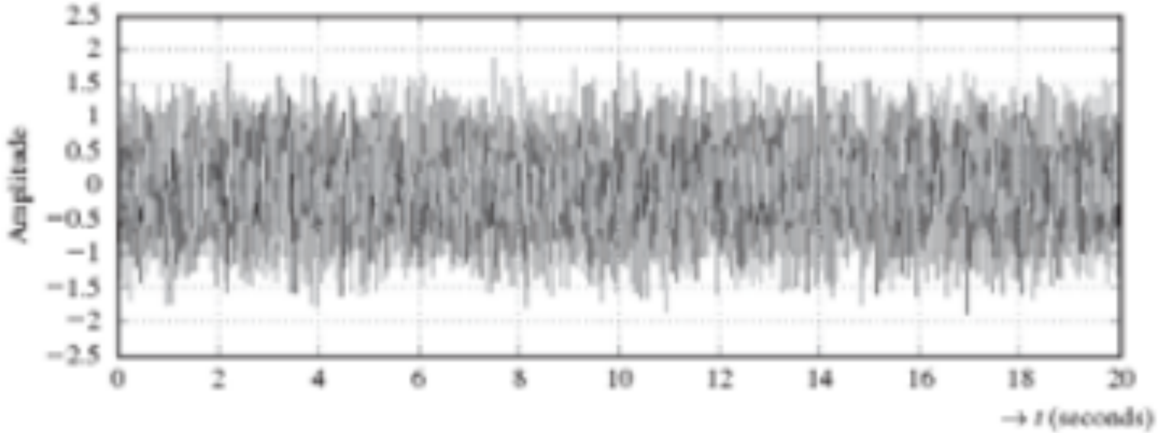
A System View

- System Classification
  - Stable (BIBO)
  - Causal (independent of future)
  - Linear (superposition principle)
  - Memory-less (no dependency on past or future)
  - Time Invariant (time shift in input→ similar time shift in output)
- We are primarily interested in LTI system

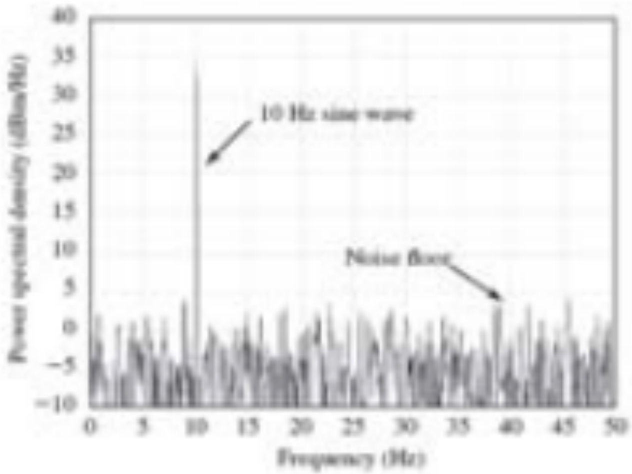
Frequency Representation

- Systems are modeled in time domain
- Often it is easier to learn about certain characteristics of a system when signals are expressed in frequency domain

Time Domain Display



Frequency Domain Display



Frequency Domain

- One way to represent a signal in frequency main is to use Fourier representation
  - Fourier Series – Periodic waveforms
  - Fourier Transform – Aperiodic waveform with finite energy (periodic signal with infinite period)
- Fourier Series can be expressed
  - Exponential FS
  - Trigonometric FS
- Expressing signals in frequency domain involves
  - magnitude & Phase

Exponential Fourier Series (Periodic Signals)

- The complex FS uses the <u>orthogonal</u> exponential function
  $$\varphi_n(t) = e^{jn\omega_0 t}$$
  where *n* is any integer, $\omega_0=2\pi/T_0$, and $T_0=(b-a)$ is the length of interval over which the orthogonal series is valid.
- A physical waveform (i.e., finite energy) may be represented over $a<t<a+T_0$
  $$\omega(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}, \text{ where}$$
  $$c_n = \frac{1}{T_0}\int_a^{a+T_0} \omega(t) e^{-jn\omega_0 t}\,dt$$
- If *w(t)* is periodic with period T_0 the series is valid over $-\infty<t<\infty$.

Notes:

- Cn is the FS coefficients
- W(t) is periodic:

$= \ldots.+C_{-1}e^{+jnwot}+C_0+C_{+1}e^{-jnwot}+\ldots..$

- **Cn is pahsor form of Spectral components**
- Cn (phasor) has phase and magnitude |Cn| & |_ Cn

Fundamental Freq. & Other Harmonics

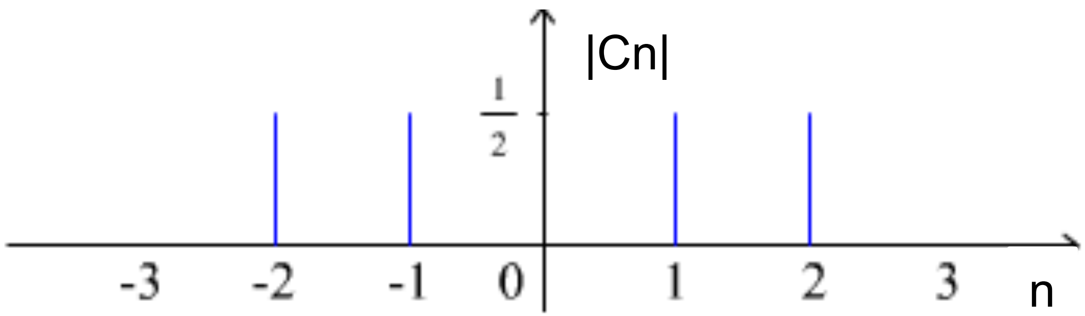
- We can represent all periodic signals as harmonic series of the form
  - C_n are the **Fourier Series Coefficients** & n is real
  - n=0 → Cn=o which is the **DC signal component**
  - n=+/-1 yields the **fundamental frequency** or the first harmonic ω_0
  - |n|>=2 harmonics

$$w(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T_0} \int_a^{a+T_0} w(t) e^{-jn\omega_0 t}\, dt$$

Fourier Series and Frequency Spectra

- We can plot the *frequency spectrum* or *line spectrum* of a signal
  - In Fourier Series n represent harmonics
  - Frequency spectrum is a graph that shows the amplitudes and/or phases of the Fourier Series coefficients *Cn*.
    - Phase spectrum ϕn
    - The lines |Cn| are called line spectra because we indicate the values by lines



Exponential Fourier Series (Properties)

$$\omega(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}, \text{ where}$$

$$c_n = \frac{1}{T_0}\int_a^{a+T_0} \omega(t) e^{-jn\omega_0 t} dt$$

- Properties of FS:
  1. If *w(t)* is real, $c_n = c_{-n}^*$
  2. If *w(t)* is real & even, $Im[c_n]=0$
  3. If *w(t)* is real & odd, $Re[c_n]=0$
  4. Pareseval theorem (Avg Pwr)

$$\frac{1}{T_0}\int_a^{a+T_0} |\omega(t)|^2 dt = \sum_{n=-\infty}^{n=\infty} |c_n|^2$$

Fourier Series (Average Power for $x_p(t)$)

Using Pareseval’s Theorem:

- The normalized power P_x of a periodic signal $x_p(t)$ is given by

$$P_x = \frac{1}{T_o}\int_{T_o} \left|x_p(t)\right|^2 dt = \frac{1}{T_o}\int_{T_o} x_p(t)x_p^*(t)dt$$

Remember:
$P_{av} = \langle V(t)^2/R \rangle$

- Substituting the FS expansion for $x_p(t)$ yields

$$P_x = \frac{1}{T_o}\int_{T_o} x_p(t)\left[\sum_{n=-\infty}^{\infty} C_n^* e^{-j2\pi n f_o t}\right]dt$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{T_o}\int_{T_o} x_p(t)e^{-j2\pi n f_o t}dt\right]C_n^*$$

$$= \sum_{n=-\infty}^{\infty} C_n C_n^* = \sum_{n=-\infty}^{\infty} \left|C_n\right|^2$$

Average power in the frequency component at $f = nf_o$

Average power of $x_p(t)$ = sum of the average power of phasor components

Different Forms of Fourier Series

- Fourier Series representation has different forms:

Note that n=k

Polar Form

Quadrature Form

Name	Equation
Exponential	$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$; $C_k = \lvert C_k \rvert e^{j\theta_k}, C_{-k} = C_k^*$
Combined trigonometric	$C_0 + \sum_{k=1}^{\infty} 2\lvert C_k \rvert \cos(k\omega_0 t + \theta_k)$
Trigonometric	$A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$
	$2C_k = A_k - jB_k, C_0 = A_0$
Coefficients	$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$

What is the relationship between them?→ Finding the coefficients!

Fourier Series in Quadrature & Polar Forms

- In quadrature form over interval $a<t<a+T_0$

$$\omega(t)=\sum_{n=0}^{n=\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{n=\infty}b_n\sin n\omega_0 t,\ \text{where}$$

$$a_n=\begin{cases}\dfrac{1}{T_0}\displaystyle\int_a^{a+T_0}\omega(t)dt\,,\ n=0\\ \dfrac{2}{T_0}\displaystyle\int_a^{a+T_0}\omega(t)\cos n\omega_0 t dt\,,\ n\geq 1\end{cases}$$

$$b_n=\frac{2}{T_0}\int_a^{a+T_0}\omega(t)\sin n\omega_0 t dt\,,\ n>1$$

Also Known as Trigonometric Form

Slightly different notations!
Note that n=k

- In polar form

$$\omega(t)=D_0+\sum_{n=0}^{n=\infty}D_n\cos(n\omega_0 t+\varphi_n),\ where$$

$$D_n=\begin{cases}a_0\,,\ n=0\\ \sqrt{a_n^2+b_n^2}\,,\ n\geq 1\end{cases}=\begin{cases}c_0\,,\ n=0\\ 2\,|\,c_n\,|,\ n\geq 1\end{cases}$$

$$\varphi_n=-\tan^{-1}\left(\frac{b_n}{a_n}\right)=\angle c_n\,,\ n\geq 1$$

$$a_n=\begin{cases}D_0\,,n=0\\ D_n\cos\varphi_n\,,n\geq 1\end{cases}$$

$$b_n=-D_n\sin\varphi_n\,,n\geq 1$$

Also Known as Combined Trigonometric Form

Important Relationships

- Euler's Relationship
  - Review Euler formulas

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$e^{j\theta} = 1\angle\theta$$

$$\arg e^{j\theta} = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right] = \theta$$

Examples of FS (A)

- Find Fourier Series Coefficients for

$$x(t) = \cos(\omega_0 t) + \sin(2\omega_0 t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

$$x(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2j}e^{j2\omega_0 t} - \frac{1}{2j}e^{-j2\omega_0 t}$$

$$C_1 = \frac{1}{2} \quad C_{-1} = \frac{1}{2} \quad C_2 = \frac{1}{2j} \quad C_{-2} = -\frac{1}{2j}$$

$C_k = 0$, all other k.

- Find Fourier Series Coefficients for

$$y(t) = \sin^2 2\omega_0 t + 2\cos\omega_0 t = \frac{1}{2}\left(1 - \cos 4\omega_0 t\right) + 2\cos\omega_0 t$$

$$y(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

$$y(t) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}e^{j4\omega_0 t} + \frac{1}{2}e^{-j4\omega_0 t}\right) + 2\left(\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right)$$

$$y(t) = \frac{1}{2} - \frac{1}{4}e^{j4\omega_0 t} - \frac{1}{4}e^{-j4\omega_0 t} + e^{j\omega_0 t} + e^{-j\omega_0 t}$$

$$C_0 = \frac{1}{2} \quad C_4 = -\frac{1}{4} \quad C_{-4} = -\frac{1}{4} \quad C_1 = 1 \quad C_{-1} = 1$$

$C_k = 0$, all other k.

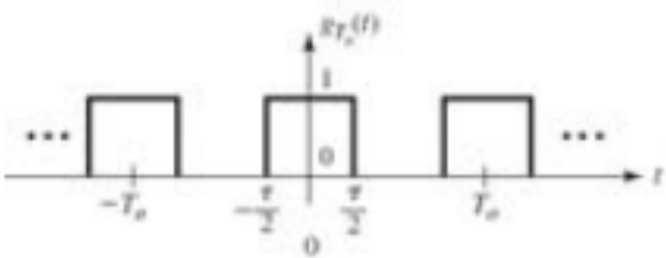
Remember:

1. $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
2. $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$
3. $\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$
4. $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
5. $\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$
6. $\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$
7. $\sin 2a = 2 \sin a \cos a$
8. $\cos^2 a = \frac{1}{2}(1 + \cos 2a)$
9. $\sin^2 a = \frac{1}{2}(1 - \cos 2a)$

Example of FS (B) (Line Spectrum of a Rectangular Pulse Train)

Determine the FS expansion of a periodic pulse train of rectangular pulses

$$g_{T_o}(t) = \sum_{n=-\infty}^{\infty} \Pi\left[\frac{(t - nT_o)}{\tau}\right]$$



- Each pulse has unity amplitude and duration τ. The FS coefficients are given by

$$C_n = \frac{1}{T_o}\int_{T_o} g_{T_o}(t) e^{-j2\pi n f_o t} dt = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} e^{-j2\pi n f_o t} dt = -\frac{1}{j2\pi n f_o T_o}\left[e^{-j\pi n f_o \tau} - e^{j\pi n f_o \tau}\right]$$

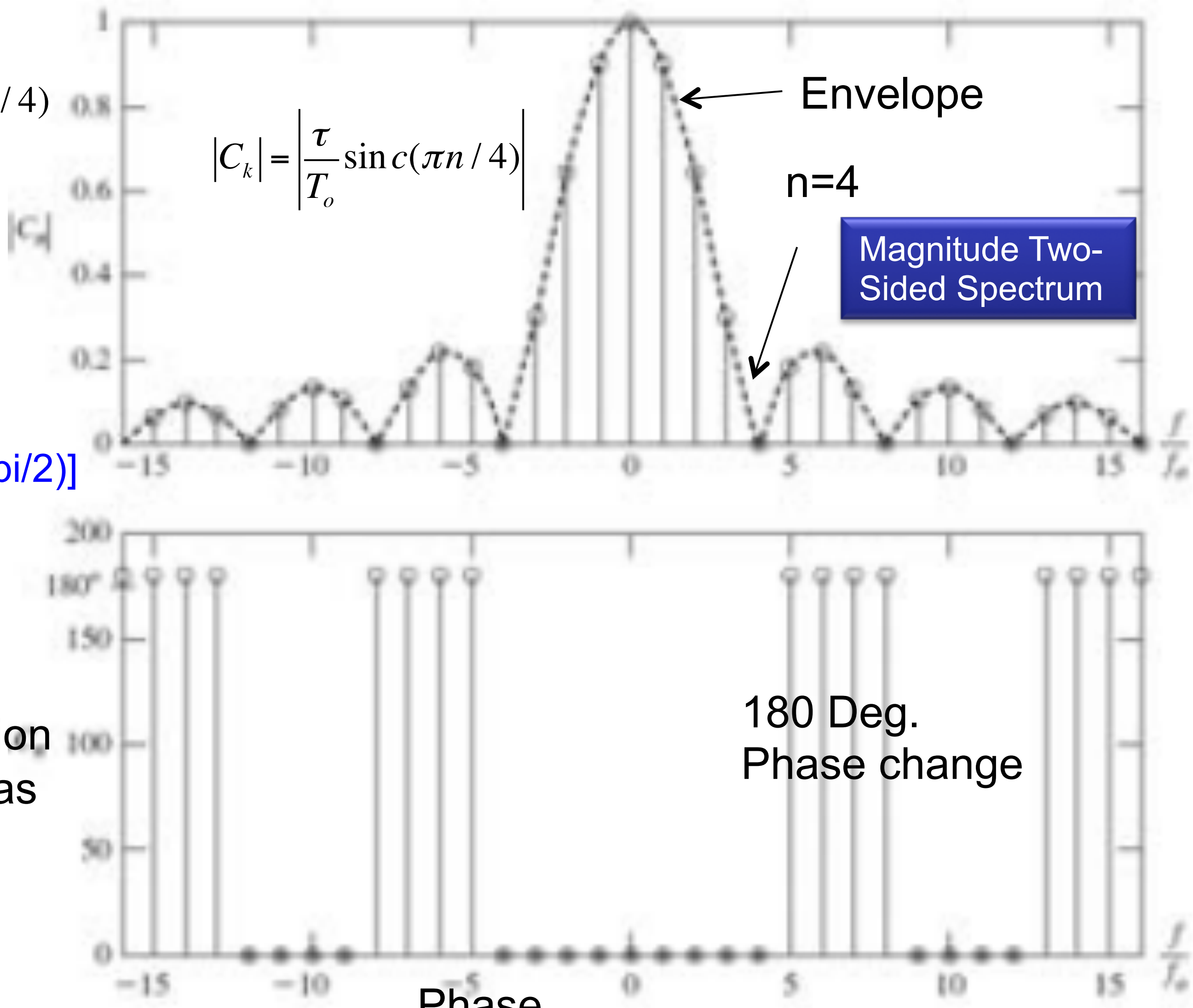
$$= \frac{\tau}{T_o}\frac{\sin(\pi n f_o \tau)}{\pi n f_o \tau}$$

See next slide

Example of FS (B-Cont.)

Note: If τ=To/4=1/4fo

$$C_k = \frac{\tau}{T_o}\operatorname{sinc}(\pi n f_o \tau) = \frac{\tau}{T_o}\operatorname{sinc}(\pi n / 4)$$



Notes:

sinc (infinity) → 0

Max value of sinc(0)=1

sinc (n.pi) = 0; n is integer>0

|Picks| occurs at sinc[n(2pi+pi/2)]

It is possible to show the Mathematical representation Of the frequency spectra as the following:

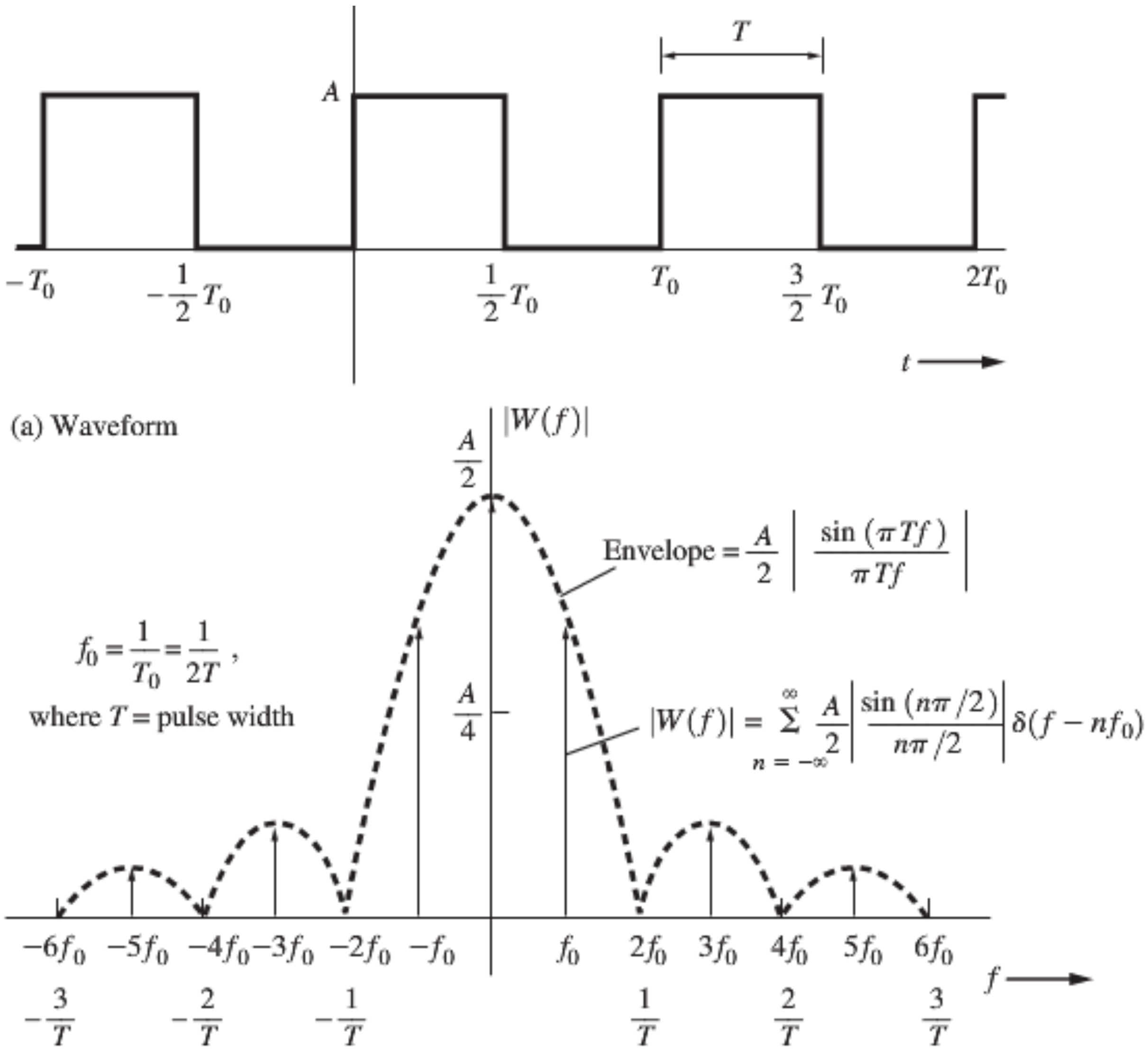
$$W(f) = \sum_{n=-\infty}^{n=\infty} c_n \, \delta(f - n f_0)$$

Phase Representation-
0, 180 deg. Change!

Applet to plot sinc function: http://mathworld.wolfram.com/SincFunction.html

Example of FS (B2)

Note that in this case there is no time-shift:

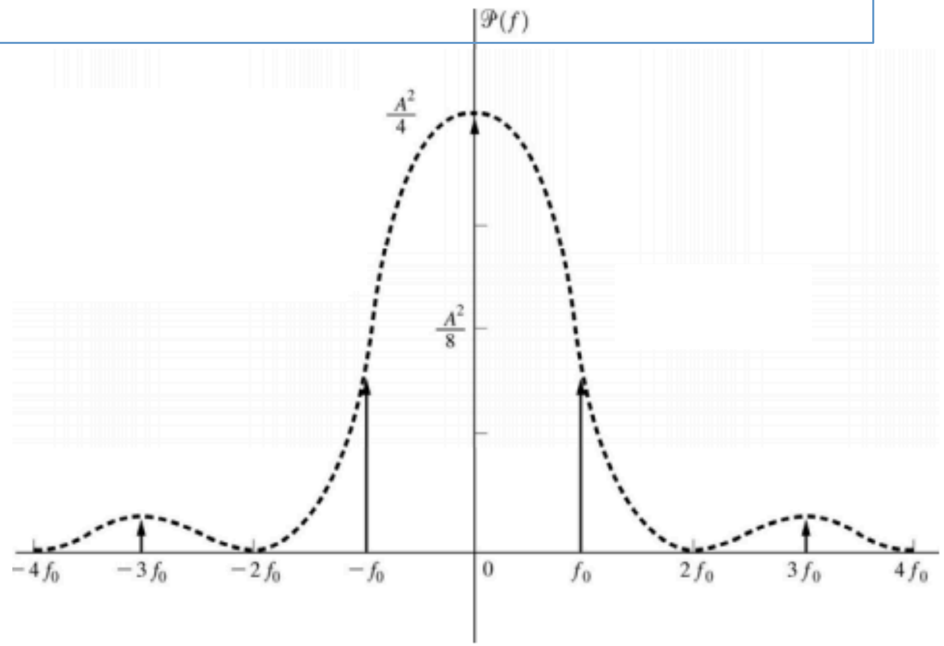


PSD of a Periodic Square Waveform

- For a periodic waveform, PSD is

$$PSD = P(f) = \sum_{n=-\infty}^{n=\infty} |c_n|^2 \, \delta(f - nf_0)$$

Therefore, $PSD = \sum_{-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$



Make sure you know the difference between Frequency Spectrum, Magnitude Frequency Spectrum, and Power Spectral Density

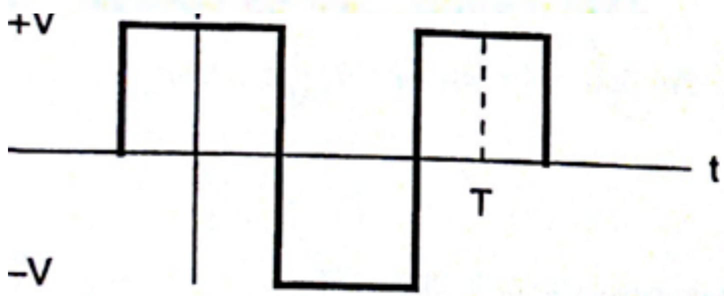
- Example: What is PSD for a square wave?

$$P(f) = \sum_{n=-\infty}^{n=\infty} \left(\frac{A}{2}\right)^2 \left(\frac{\sin(n\pi/2)}{n\pi/2}\right)^2 \delta(f - nf_0)$$

Using Example of FS (B2)

Example of FS (C) – A different Approach

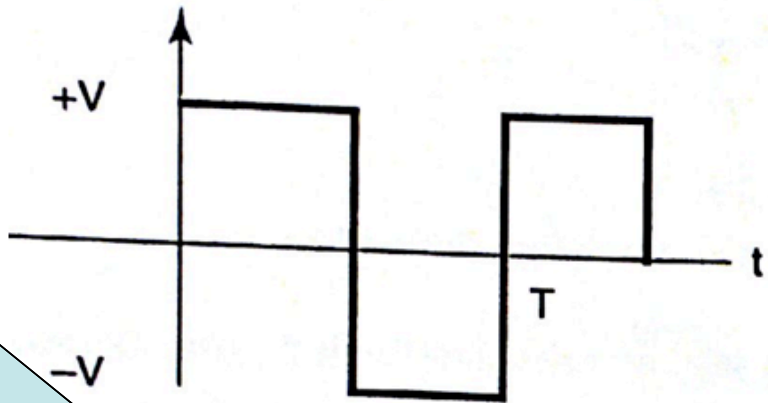
- Note that here we are using quadrature form of amplitude shifted version of v(t): $\omega(t) = \sum_{n=0}^{n=\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{n=\infty} b_n \sin n\omega_0 t,$



Even function

$$v(t) = \frac{4V}{\pi}\cos \omega t - \frac{4V}{3\pi}\cos 3\omega t + \frac{4V}{5\pi}\cos 5\omega t + \cdots$$

$$v_{sqr\_bipolar\_even}(t) = \sum_{N=odd}^{\infty} 2V \sin c(N\pi/2)\cos(N\omega t)$$



Odd function

$$v(t) = \frac{4V}{\pi}\sin \omega t + \frac{4V}{3\pi}\sin 3\omega t + \cdots$$

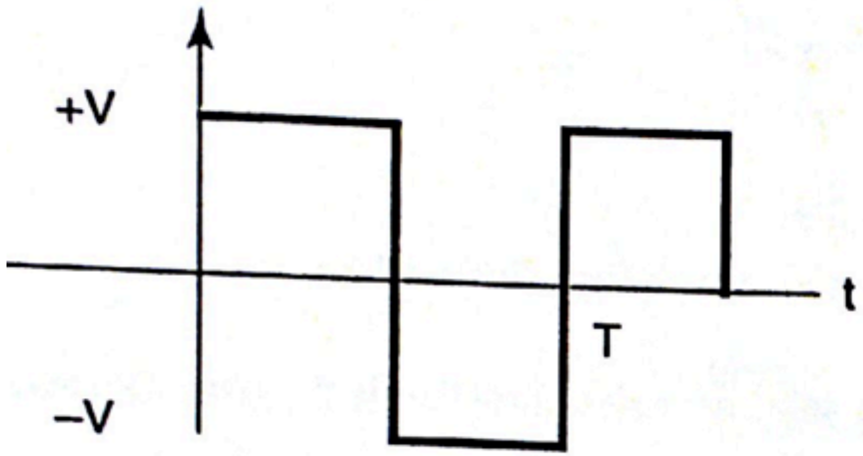
$$v(t) = \sum_{N=odd}^{\infty} \frac{4V}{N\pi}\sin N\omega t$$

Do it!

Note that N=n; T=To

A Closer Look at the Quadrature Form of FS

- Consider the following quadrature FS representation of an odd square waveform with no offset:

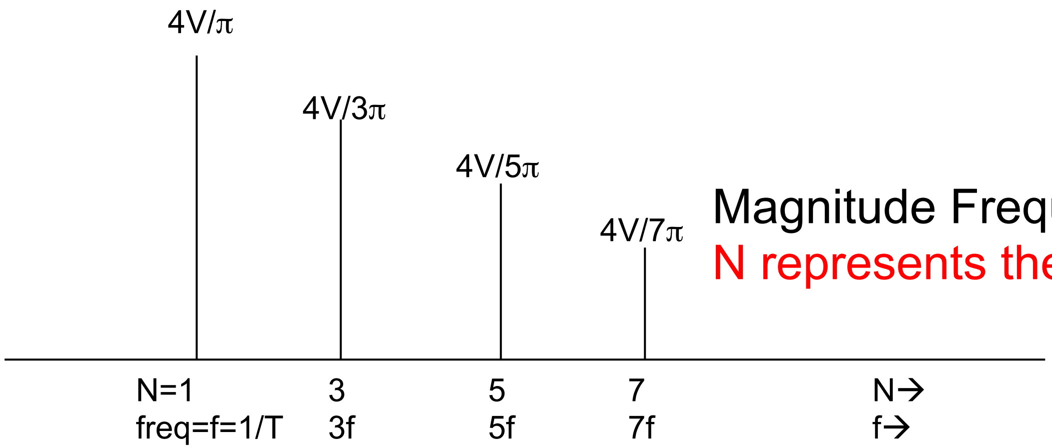


$$v(t) = \frac{4V}{\pi}\sin \omega t + \frac{4V}{3\pi}\sin 3\omega t + \cdots$$

$$v(t) = \sum_{N=\text{odd}}^{\infty} \frac{4V}{N\pi}\sin N\omega t$$

$$W(f) = \sum_{n=-\infty}^{n=\infty} c_n\, \delta(f - nf_0)$$

Thus: $C_n = 4V/N\pi$

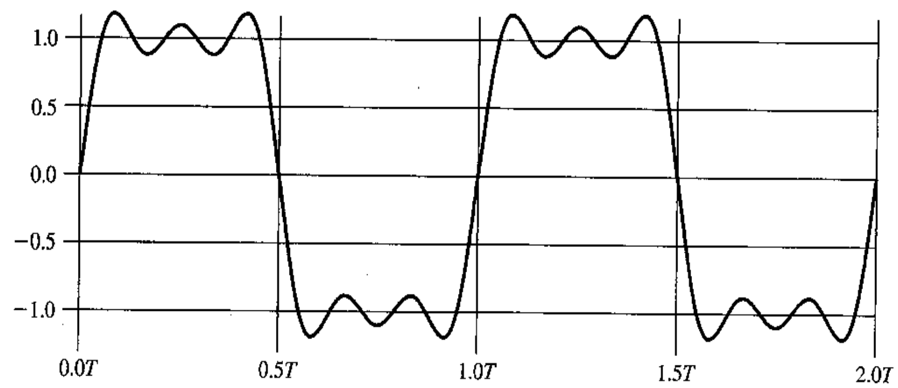


Magnitude Frequency Spectrum W(f)=FS[v(t)]

N represents the Harmonic NUMBER

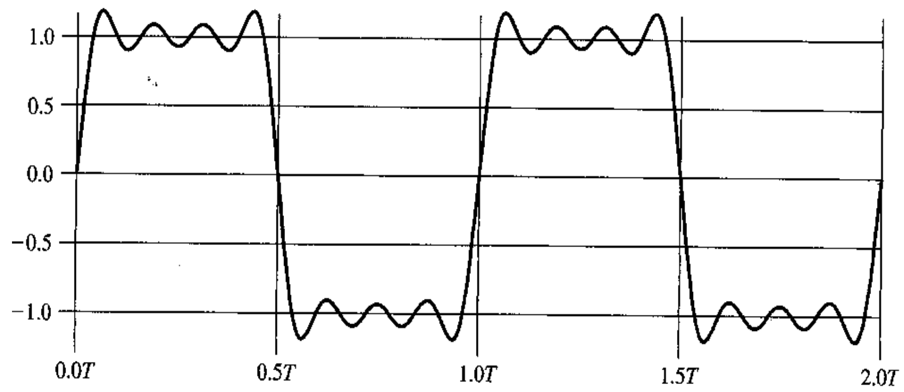
So 3f represents the third harmonic number

Generating an Square Wave



(a) $(4/\pi)\,[\sin(2\pi f t) + (1/3)\sin(2\pi(3f)t) + (1/5)\sin(2\pi(5f)t)]$

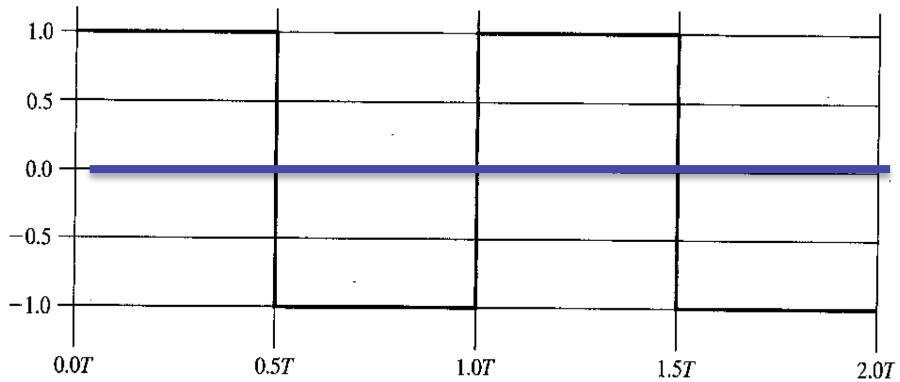
N=1,3,5



(b) $(4/\pi)\,[\sin(2\pi f t) + (1/3)\sin(2\pi(3f)t) + (1/5)\sin(2\pi(5f)t) + (1/7)\sin(2\pi(7f)t)]$

N=1,3,5, 7

This is how the time-domain waveform of the first 7 harmonics looks like!



(c) $(4/\pi)\,\Sigma\,(1/k)\sin(2\pi(kf)t)$, for k odd

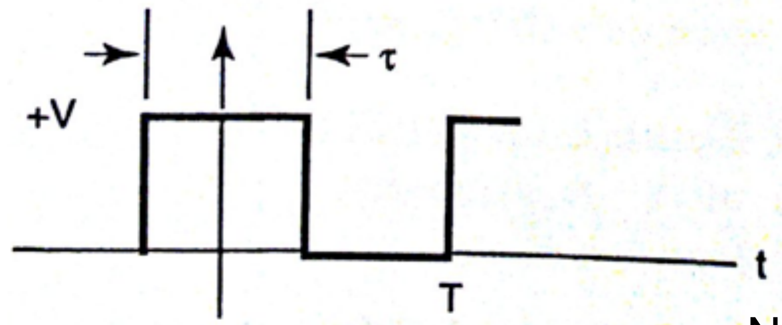
N=1,3,5, 7, 9, …..

Frequency Components of Square Wave

$$v(t) = \sum_{N=\text{odd}}^{\infty} \frac{4V}{N\pi} \sin N\omega t$$

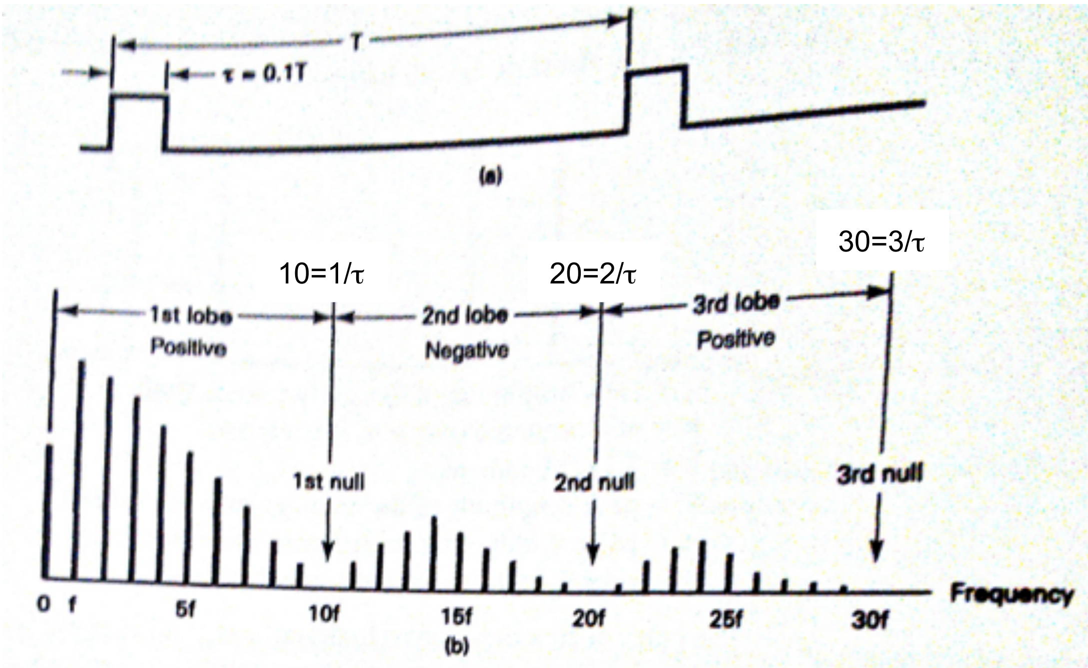
Fourier Expansion

What Is the FS of A Pulse Signal?

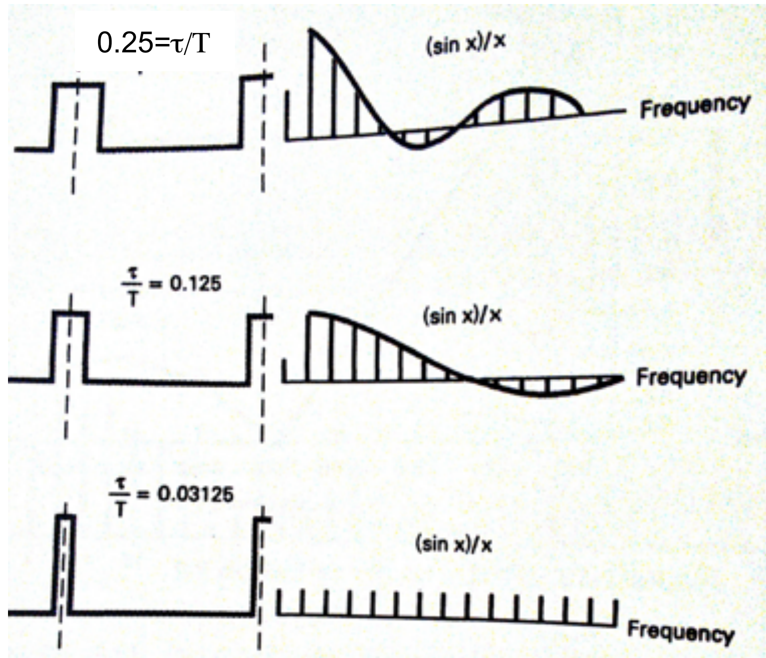


$$v(t) = \sum_{N=1}^{\infty} \frac{2V\tau}{T_o} \sin c\left(\frac{N 2\pi f_o}{T_o}\right) \cos(N\pi t)$$

Note that the width of the pulse can change!



Magnitude Line Spectra of the pulse signal – note that the envelope is a sinc ( ) function!



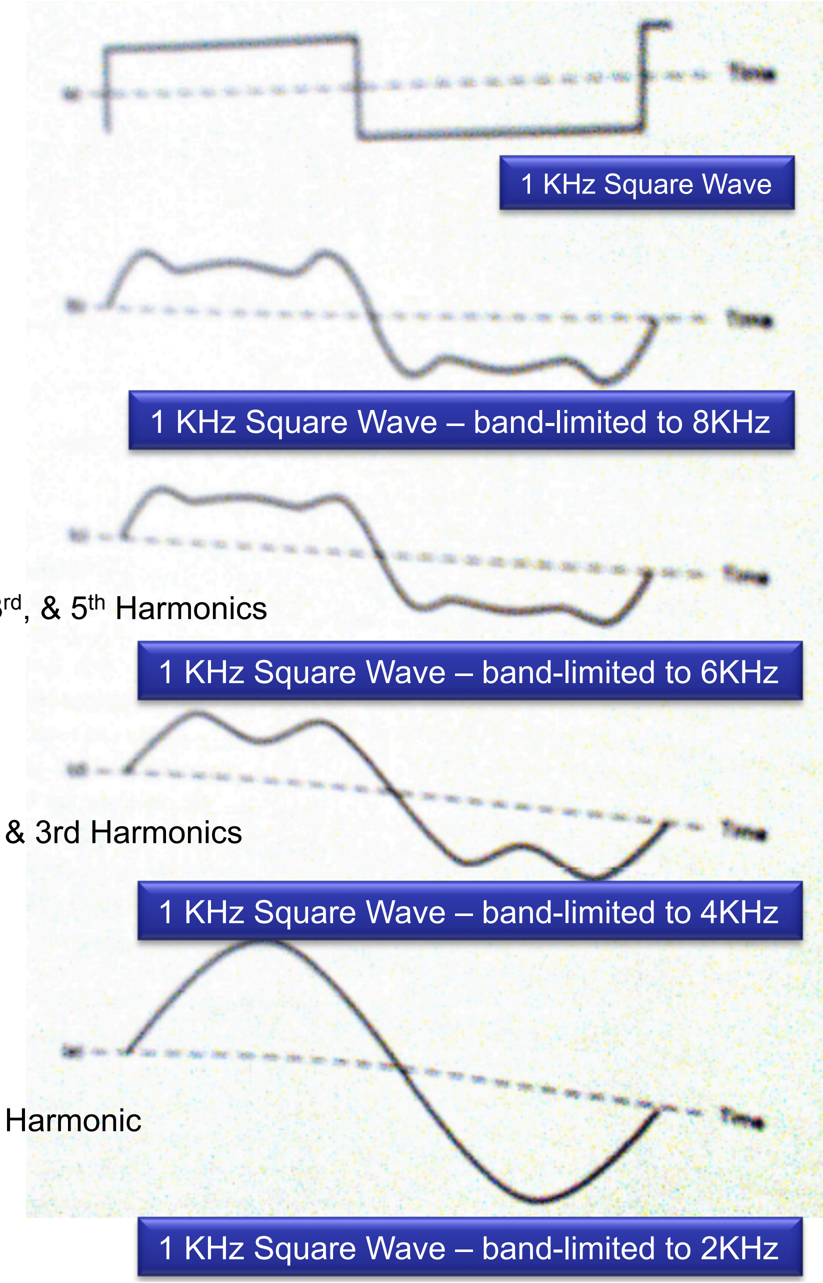
What happens to the envelope as the pulse gets smaller?

Bandlimiting Effects on Signals

- All communication systems have some finite bandwidth
- Sufficient BW must be guaranteed to reserve the signal integrity

$$v(t) = \sum_{N=odd}^{\infty} V \sin c(N\pi/2)\cos(N\omega t)$$

$$W(f) = \sum_{n=-\infty}^{n=\infty} c_n \, \delta(f - nf_0)$$



1st, 3rd, & 5th Harmonics

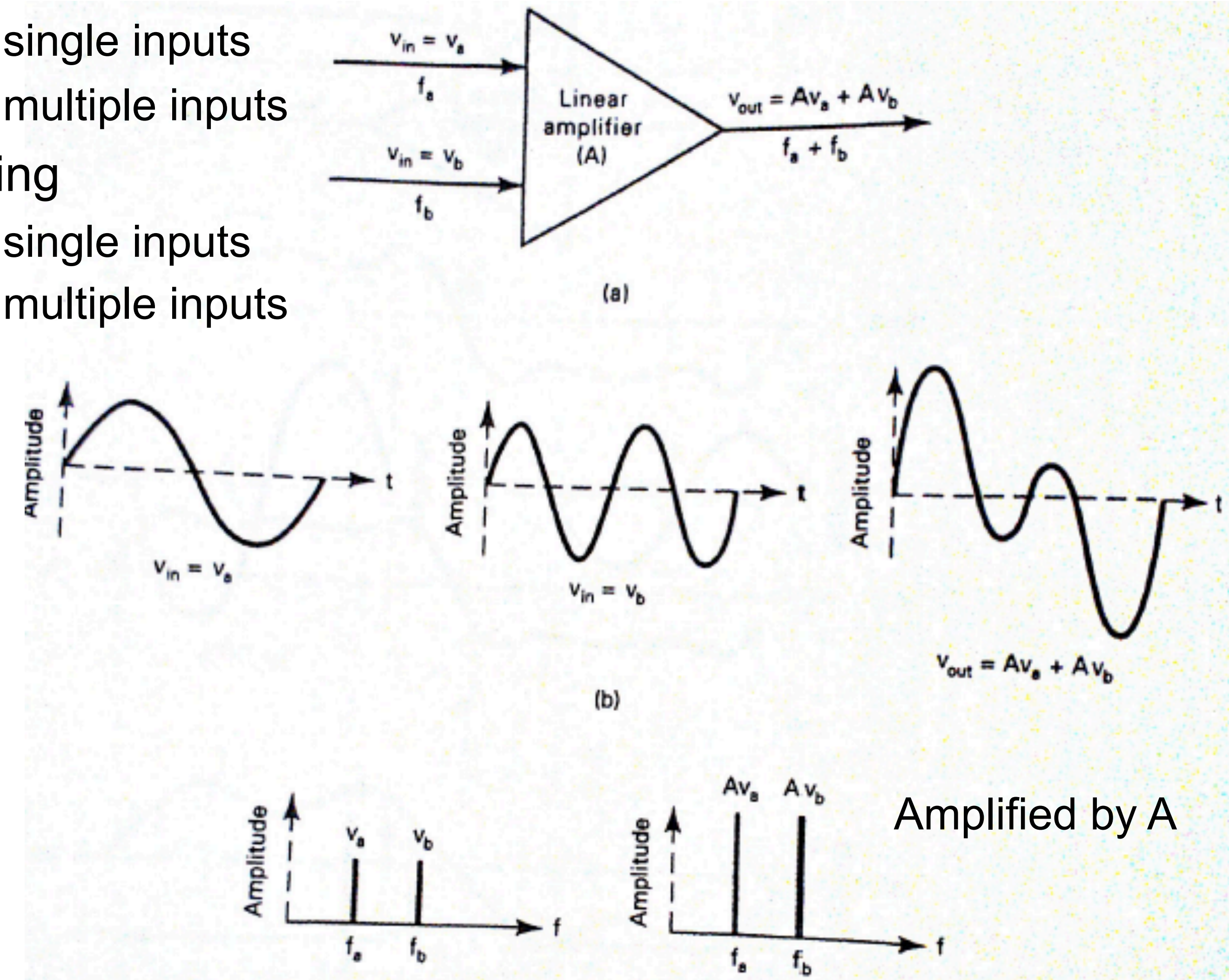
First & 3rd Harmonics

First Harmonic

A waveform w(t) is said to be (absolutely) bandlimited to B hertz if
W(f) = F[w(t)] = 0, for |fo| >or = B

Bandlimiting in Mixing Devices

- Mixing is the process of combining two or more signals (e.g., Op-Amps)
  - Linear Summing
    - Amplifiers with single inputs
    - Amplifiers with multiple inputs
  - Nonlinear Summing
    - Amplifiers with single inputs
    - Amplifiers with multiple inputs



Amplified by A

Bandlimiting in Mixing Devices

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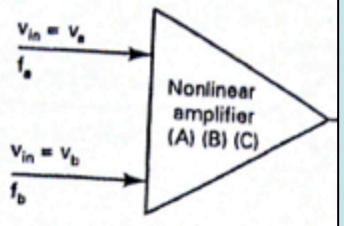


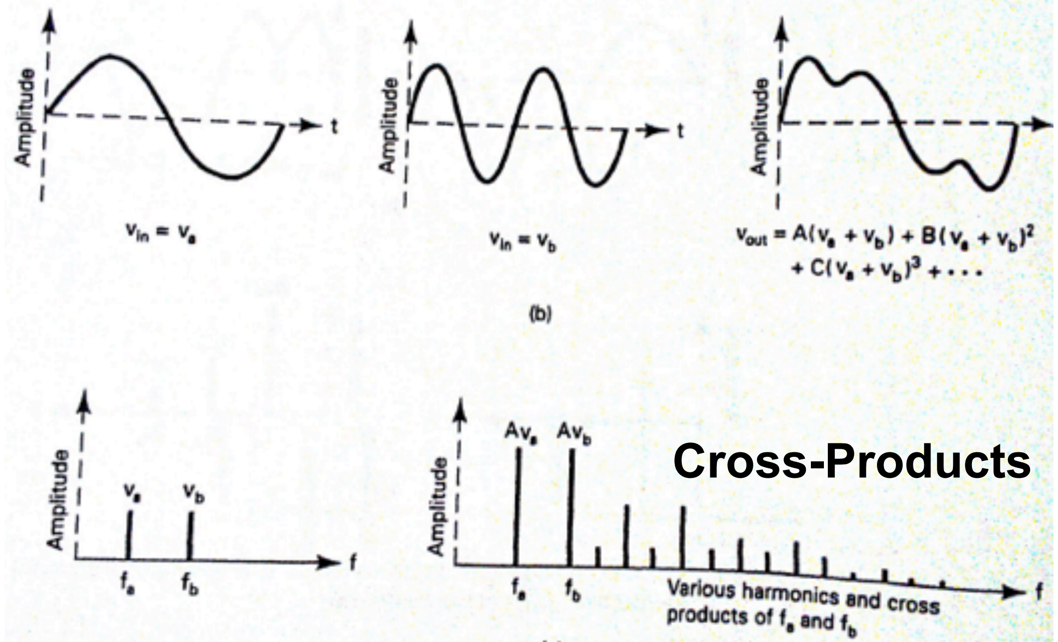
$$v_{out} = Av_{in} + Bv_{in}^2 + Cv_{in}^3$$
$$v_{in} = V_a \sin(2\pi f_a t) + V_b \sin(2\pi f_b t)$$
$$--> v_{out} = A(V_a \sin(2\pi f_a t) + V_b \sin(2\pi f_b t))$$
$$+B(V_a \sin(2\pi f_a t) + V_b \sin(2\pi f_b t))^2$$
$$+C(V_a \sin(2\pi f_a t) + V_b \sin(2\pi f_b t))^3$$

For nonlinear case an infinite number of harmonic frequencies are produced!

If these cross-products are undesired → we call them intermodulation distortion!

If these cross-products are desired → we call them modulation!

Cross-Products = m.f$_a$ +/- n.f$_b$



Example

- Assume we have a nonlinear system receiving two tones with frequencies of 5KHz and 7 KHz. Plot the output frequency spectrum for the first three harmonics (assume m & n can each be 1 & 2).

  - **Fundamental frequencies (first harmonic): 5KHz & 7KHz**
  - **Harmonics:**
    - **Second harmonic: 10KHz & 14KHz**
    - **Third harmonic: 15KHz & 21KHz**
  - **Cross-Products = $m.f_a$ +/- $n.f_b$**
    - **n=1 & m=1 → 5+/-7=12KHz & 2KHz**
    - **n=1 & m=2 → 5+/-14=9KHz & 19KHz**
    - **n=2 & m=1 → 10+/-7=3KHz & 17KHz**
    - **n=2 & m=2 → 10+/-14=24KHz & 4KHz**

All together there are 14 frequencies on the frequency spectrum!

Exercises Related to FS

- Review Schaum's Outline Chapter 1

Fourier Transform (1)

Remember: Fourier Series:

$$\omega(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}, \text{ where}$$

$$c_n = \frac{1}{T_0} \int_a^{a+T_0} \omega(t) e^{-jn\omega_0 t} dt$$

- How can we represent a waveform?
  - Time domain
  - Frequency domain → rate of occurrences
- Fourier Transform (FT) is a mechanism that can find the frequencies *w(t)*:

$$W(f) = \mathcal{F}[w(t)] = \int_{-\infty}^{\infty} [w(t)] e^{-j2\pi f t} dt$$

- W(f) is the two-sided spectrum of w(t) → positive/neg. freq.
- W(f) is a complex function:

$$W(f) = X(f) + jY(f) = |W(f)| e^{j\theta(f)} = \sqrt{X^2(f) + Y^2(f)}, \theta(f) = \tan^{-1}\left(\frac{Y(f)}{X(f)}\right)$$

(Quadrature Components: $X(f) + jY(f)$; Phasor Components: $|W(f)| e^{j\theta(f)}$)

- Time waveform can be obtained from spectrum using Inverse FT

$$w(t) = \int_{-\infty}^{\infty} W(f) e^{j2\pi f t} df$$

Fourier Transform (2)

- Thus, Fourier Transfer Pair: w(t) ←→ W(f)
- W(t) is Fourier transformable if it satisfies the **Dirichlet** conditions (sufficient conditions):
  - Over a finite time interval w(t), is single valued with a finite number of Max & Min, & discontinuities.

$$E = normalized\ \ energy = \int_{-\infty}^{\infty} |\,\omega(t)\,|^2\ dt < \infty$$

* dir-i-kley

Dirac Delta and Unit Step Functions

1. Dirac Delta Function (Unit impulse)

$$\int_{-\infty}^{\infty} \omega(x)\delta(x)dx = \omega(0)$$

where w(x) is continuous at $x=0$.

- Alternative definitions:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1, \quad \delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

- Shifting Property of Delta Function

$$\int_{-\infty}^{\infty} \omega(x)\delta(x - x_0)dx = \omega(x_0)$$

2. Unit step function

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

Note that

$$\int_{-\infty}^{t} \delta(x)dx = u(t), \text{ thus } \frac{du(t)}{dt} = \delta(t)$$

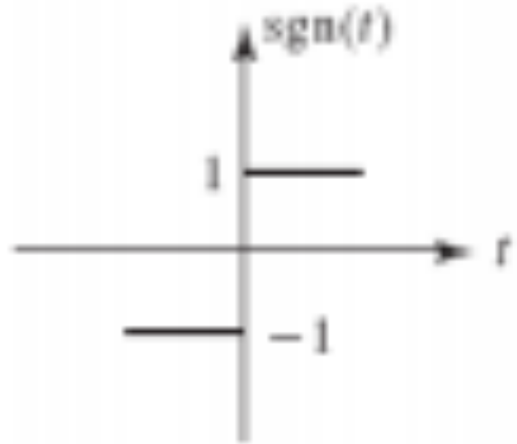
* dih-**rak**

FT of Signum Functions

- The signun signal sgn(t) can be expressed as

$$\mathrm{sgn}(t) = \begin{cases} 1, & t \geq 0 \\ -1, & t \leq 0 \end{cases}$$

$$= \lim_{\alpha \to 0} \begin{cases} e^{-\alpha t}, & t \geq 0 \\ e^{\alpha t}, & t \leq 0 \end{cases}$$



- The FT of sgn(t) is given by

$$\Im\{\mathrm{sgn}(t)\} = \lim_{\alpha \to 0}\left[\int_{-\infty}^{0} e^{\alpha t} e^{-j2\pi f t} dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j2\pi f t} dt\right]$$

$$= \lim_{\alpha \to 0}\left[\int_{-\infty}^{0} e^{(\alpha - j2\pi f)t} dt + \int_{0}^{\infty} e^{-(\alpha + j2\pi f)t} dt\right]$$

$$= \lim_{\alpha \to 0} \frac{-4j\pi f}{\alpha^2 + 4\pi^2 f^2} = \frac{1}{j\pi f}$$

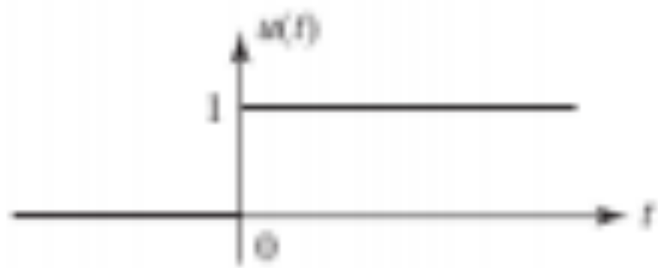
FT of Unit Step

- The unit step function $u(t)$ can be expressed as

$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t)$$

- Taking the FT of both sides yields

$$U(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$



FT Examples (1)

1. Find FT of ***impulse delta signal.***

$$F\{\delta(t)\} = D(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^0 = 1$$

Note that in general:

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$$

In our case, to = 0 and f(to) =1

2. Find FT of a ***DC waveform*** $\omega(t) = 1$

$$F\{1\} = \int_{-\infty}^{\infty} e^{-j\omega t} dt = \delta(\omega)$$

This can be shown by taking the inverse of delta function.

$$F^{-1}\{\delta(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} dt = e^0 = 1,$$

See Appendix A of the Textbook!

3. Find the spectrum of an exponential pulse.

$$\omega(t) = \begin{cases} e^{-t}, t > 0 \\ 0, t < 0 \end{cases}$$

$$W(f) = \int_{0}^{\infty} e^{-t} e^{-j\omega t} dt = \left. \frac{-e^{-(1+j\omega)t}}{(1+j\omega)} \right|_0^{\infty} = \frac{1}{(1+j\omega)}$$

The quadrature components are:

$$X(f) = \frac{1}{1 + (2\pi f)^2} \quad \text{and} \quad Y(f) = \frac{-2\pi f}{1 + (2\pi f)^2}$$

The polar components are:

$$|W(f)| = \sqrt{\frac{1}{1 + (2\pi f)^2}} \quad \text{and} \quad \theta(f) = -\tan^{-1}(2\pi f)$$

Pay attention!

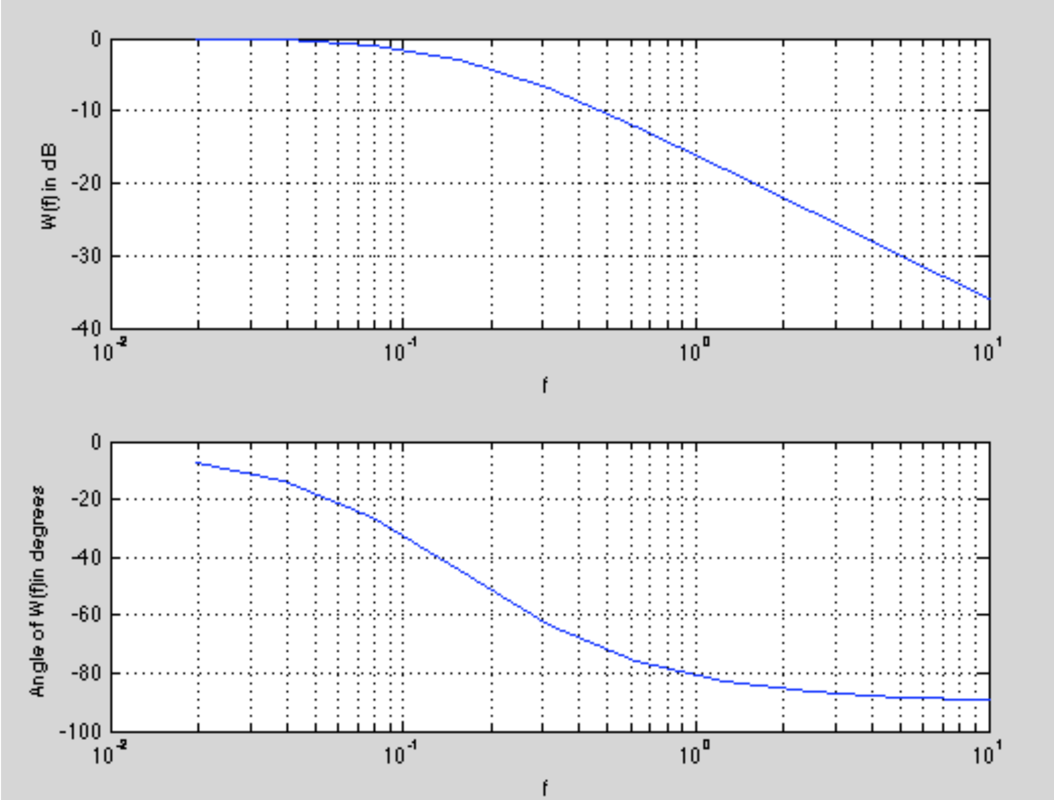
NEXT→

FT Example (2)

```
% The Magnitude-Phase Spectral Functions
% will be plotted.
% The Magnitude function will be plotted in dB units.
% The Phase function will be plotted in degree units.

clear;

for (k = 1:10)
  f(k) = 10*2^(-10)*2^k;
  W(k) = 1/(1 + 2*pi*f(k)*sqrt(-1));
end;

B = log(W);
WdB = (20/log(10))*real(B);
Theta = 180/pi*imag(B);
subplot(211);
semilogx(f,WdB);
xlabel('f');
ylabel('W(f) in dB');
grid;

subplot(212);
semilogx(f,Theta);
xlabel('f');
ylabel('Angle of W(f)in degrees');
grid;
subplot(111);
```

Magnitude-Phase Form:

$$|W(f)| = \sqrt{\frac{1}{1 + (2\pi f)^2}} \quad \text{and} \quad \theta(f) = -\tan^{-1}(2\pi f)$$



Note: Pay attention to how the equations are setup!

Phase Difference & Time Delay

What does time delay have to do with phase angle?

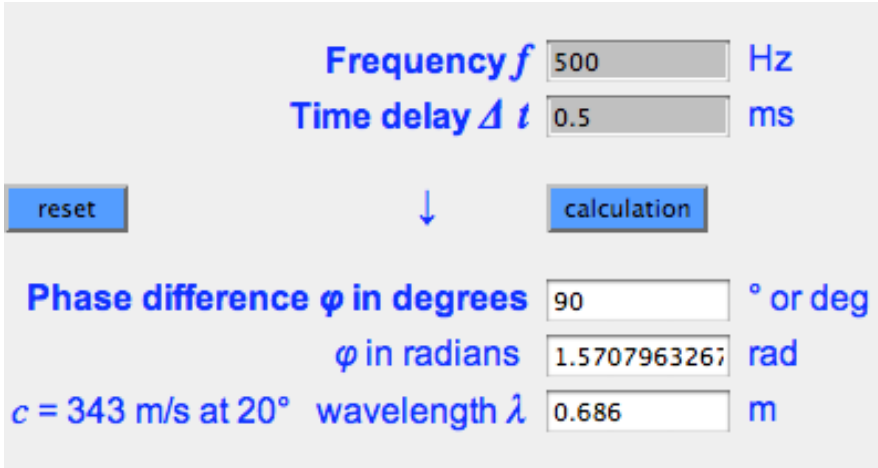
Calculation between phase angle $\varphi°$ in degrees (deg), the time delay Δt and the frequency f is:

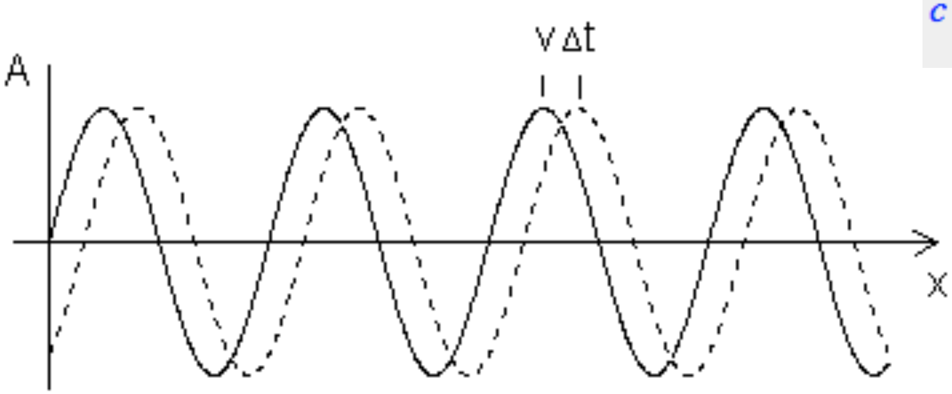
Phase angle (deg) $\varphi° = 360° \cdot f \cdot \Delta t$

(Time shift) Time difference $\Delta t = \dfrac{\varphi°}{360 \cdot f}$

Frequency $f = \dfrac{\varphi°}{360 \cdot \Delta t}$

$\lambda = c / f$ and $c = 343$ m/s at 20°C.





Other FT Properties

Operation	Function	Fourier Transform
Linearity	$a_1 w_1(t) + a_2 w_2(t)$	$a_1 W_1(f) + a_2 W_2(f)$
Time delay	$w(t - T_d)$	$W(f) e^{-j\omega T_d}$
Scale change	$w(at)$	$\frac{1}{\lvert a \rvert} W\left(\frac{f}{a}\right)$
Conjugation	$w^*(t)$	$W^*(-f)$
Duality	$W(t)$	$w(-f)$
Real signal frequency translation [$w(t)$ is real]	$w(t)\cos(w_c t + \theta)$	$\frac{1}{2}\left[e^{j\theta} W(f - f_c) + e^{-j\theta} W(f + f_c)\right]$
Complex signal frequency translation	$w(t) e^{j\omega_c t}$	$W(f - f_c)$
Bandpass signal	$\mathrm{Re}\{g(t) e^{j\omega_c t}\}$	$\frac{1}{2}\left[G(f - f_c) + G^*(-f - f_c)\right]$
Differentiation	$\frac{d^n w(t)}{dt^n}$	$(j2\pi f)^n W(f)$

Find FT of $w(t)\sin(w_c t)$!

$w(t)\sin(w_c t)$ = w(t)*(cos(wct-90)) = ½ [e^-90] W(f-fc) + [e^+90] W(f+fc]=½ [-j] W(f-fc) +[j] W(f+fc]

Spectrum of A Sinusoid

- Given v(t) = $A\sin(w_o t)$ the following function plot the magnitude spectrum and phase Spectrum of v(t): |v(f)| & θ(f)

$$v(t) = A\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right)$$

$$V(f) = \int_{-\infty}^{\infty} A\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) e^{-j\omega t} dt$$

$$= \frac{A}{2j}\int_{-\infty}^{\infty} e^{-2j\pi(f-f_0)t} dt - \frac{A}{2j}\int_{-\infty}^{\infty} e^{-2j\pi(f+f_0)t} dt$$

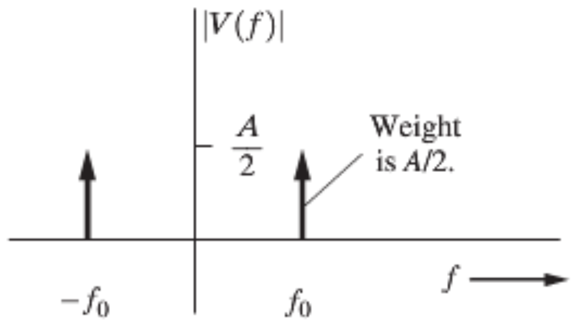
$$= j\frac{A}{2}[\delta(f+f_0) - \delta(f-f_0)]$$

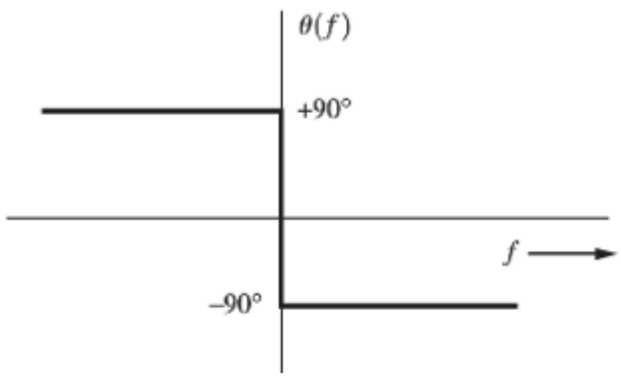
Similar to FT for DC waveform Example

- The magnitude spectrum is

$$|V(f)| = \frac{A}{2}\delta(f-f_0) + \frac{A}{2}\delta(f+f_0)$$

Note that V(f) is purely imaginary
→ When f>0, then θ(f)= -π/2
→ When f<0, then θ(f)= +π/2

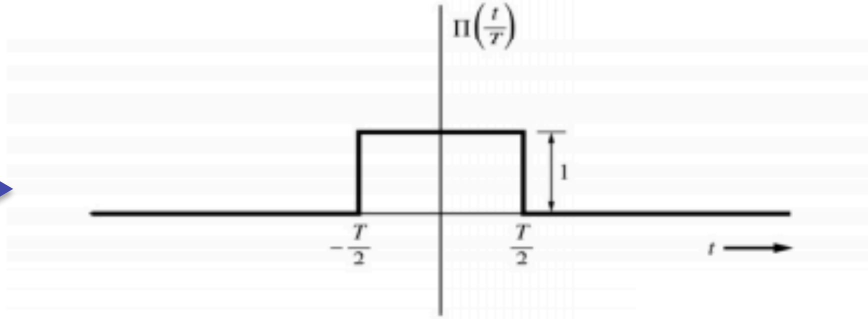




Other Fourier Transform Pairs (1)

- **Rectangular pulse:**

$$\Pi\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| \le \dfrac{T}{2} \\ 0, & |t| > \dfrac{T}{2} \end{cases}$$

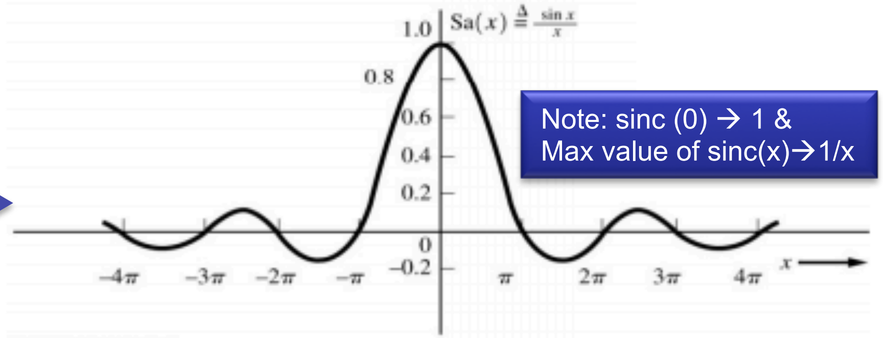


(a) Rectangular Pulse

Spectrum of a rectangular pulse

$$W(f) = \int_{-T/2}^{T/2} 1 e^{-j\omega t} dt = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$= T\frac{\sin(\omega T/2)}{(\omega T/2)} = TSa(\pi T f)$$



(b) Sa(x) Function

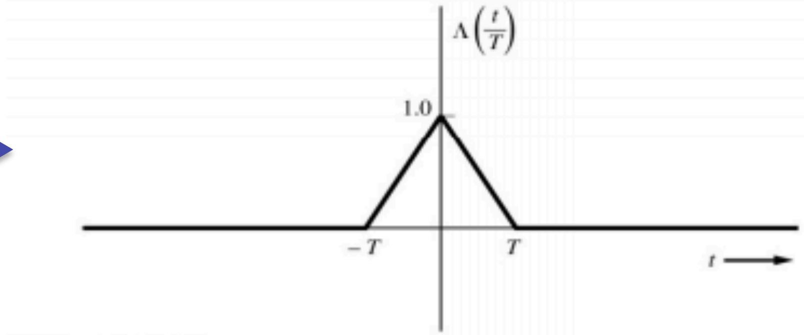
- **Sa (or Sinc) function:**

$$Sa(x) = \frac{\sin(x)}{x} = \text{Sinc}\,(x/\pi)$$

- **Triangular function:**

$$Tri(t) = \Lambda\left(\frac{1}{T}\right) = \begin{cases} 1 - \dfrac{|t|}{T}, & |t| \le T \\ 0, & |t| > T \end{cases}$$

$$Tri(t) = \Lambda\left(\frac{t}{T}\right) \leftrightarrow T \cdot Sa^2(\pi f T)$$



(c) Triangular Function

Sa stands for **Sampling Function**

Other Fourier Transform Pairs (2)

$$\Pi\left(\frac{t}{T}\right) \leftrightarrow T \cdot Sa(\pi T f)$$

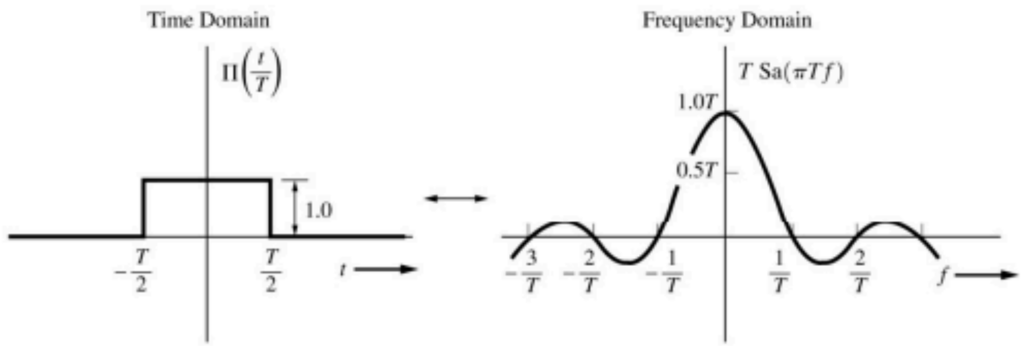
it is convenient to represent binary 1 & 0, e.g., in TTL logic circuits by the pulse.

$$2W \cdot Sa(2\pi W t) \leftrightarrow \Pi\left(\frac{f}{2W}\right)$$

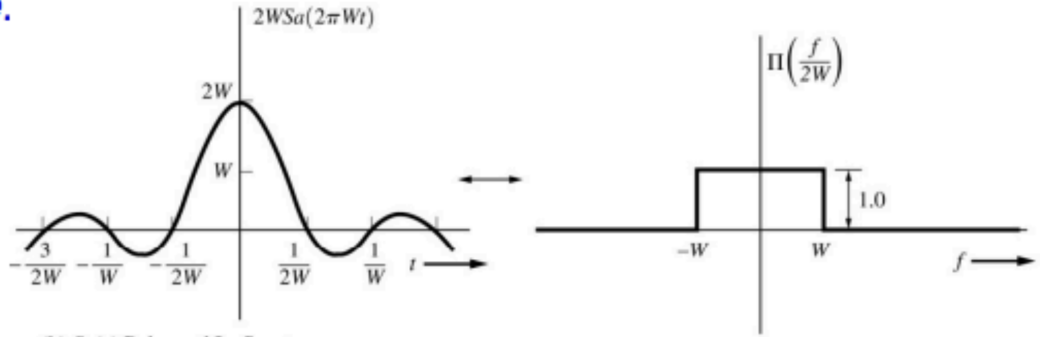
Using Duality Property

$$\omega(t) = \Lambda\left(\frac{t}{T}\right) \leftrightarrow T\,\frac{[\sin(\pi f T)]^2}{(\pi f T)^2}, or$$

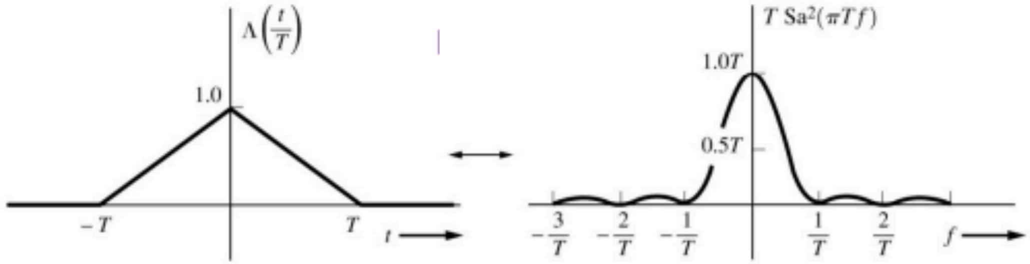
$$\omega(t) = \Lambda\left(\frac{t}{T}\right) \leftrightarrow T \cdot Sa^2(\pi f T)$$



(a) Rectangular Pulse and Its Spectrum



(b) $Sa(x)$ Pulse and Its Spectrum



(c) Triangular Pulse and Its Spectrum

Note: $\omega(t) = \Lambda\left(\frac{t}{T}\right) = \begin{cases} 1 - t/T, & 0 < t < T \\ 0, & t > T \\ 1 + t/T, & -T < t < 0 \\ 0, & t < -T \end{cases}$

Other Fourier Transform Pairs (3)

Function	Time Waveform $w(t)$	Spectrum $W(f)$
Rectangular	$\Pi\left(\frac{t}{T}\right)$	$T[\mathrm{Sa}(\pi f T)]$
Triangular	$\Lambda\left(\frac{t}{T}\right)$	$T[\mathrm{Sa}(\pi f T)]^2$
Unit step	$u(t) \triangleq \begin{cases} +1, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
Signum	$\mathrm{sgn}(t) \triangleq \begin{cases} +1, & t > 0 \\ -1, & t < 0 \end{cases}$	$\frac{1}{j\pi f}$
Constant	1	$\delta(f)$
Impulse at $t = t_0$	$\delta(t - t_0)$	$e^{-j2\pi f t_0}$
Sin c	$\mathrm{Sa}(2\pi W t)$	$\frac{1}{2W}\Pi\left(\frac{f}{2W}\right)$
Phasor	$e^{j(\omega_0 t + \varphi)}$	$e^{j\varphi}\delta(f - f_0)$
Sinusoid	$\cos(\omega_c t + \varphi)$	$\frac{1}{2}e^{j\varphi}\delta(f - f_c) + \frac{1}{2}e^{-j\varphi}\delta(f + f_c)$
Gaussian	$e^{-\pi(t/t_0)^2}$	$t_0 e^{-\pi(f t_0)^2}$
Exponential, one-sided	$\begin{cases} e^{-t/T}, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{T}{1 + j2\pi f T}$

Examples

1. Using superposition, find the spectrum for a waveform

$$\omega(t) = \Pi\left(\frac{t-5}{10}\right) + 8\sin(6\pi t)$$

Solution: Use rectangular & scaling

$$F\left[\Pi\left(\frac{t-5}{10}\right)\right] = 10\frac{\sin(10\pi f)}{(10\pi f)}e^{-j2\pi f5}$$

Using time delay property

For 8sin(6πt), we have:

Note: 2πfo=2π(3)

$$F[8\sin(6\pi t)] = j\frac{8}{2}[\delta(f+3) - \delta(f-3)]$$

Therefore

$$W(f) = 10\frac{\sin(10\pi f)}{10\pi f}e^{-j10\pi f} + j4[\delta(f+3) - \delta(f-3)]$$

2. Using integration, find the spectrum of

$$\omega(t) = 5 - 5e^{-2t}u(t)$$

Solution:

$$W(f) = \int_{-\infty}^{\infty} \omega(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} 5e^{-j2\pi ft}dt - 5\int_{-\infty}^{\infty} e^{-2t}e^{-j2\pi ft}u(t)dt$$

$$= 5\delta(f) - 5\left.\frac{e^{-2(1+j\pi f)t}}{-2(1+\pi f)}\right|_0^{\infty}, or$$

$$W(f) = 5\delta(f) - \frac{2.5}{1+j\pi f}$$

For what freq. W(f) has its max?

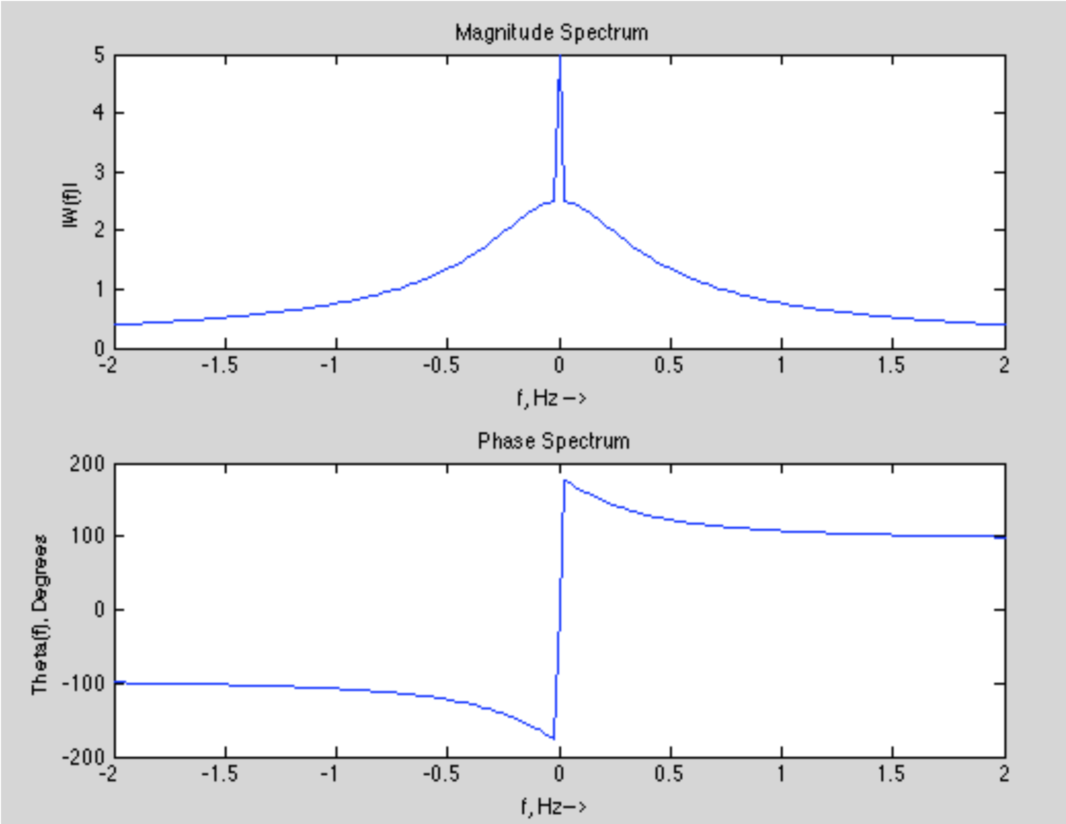
See the Gaussian Exponential One-sided Property! (T=1/2)

Plotting Magnitude and Phase Spectrum

```matlab
% Continuous part of Spectrum
Wmag = zeros(length(f),1);
Theta = zeros(length(f),1);
for (i=1:1:length(f))
   Wmag(i) = abs(-5/(2+2j*pi*f(i)));
   Theta(i)=(180/pi)* angle(-5/(2+2j*pi*f(i)));
end;
```

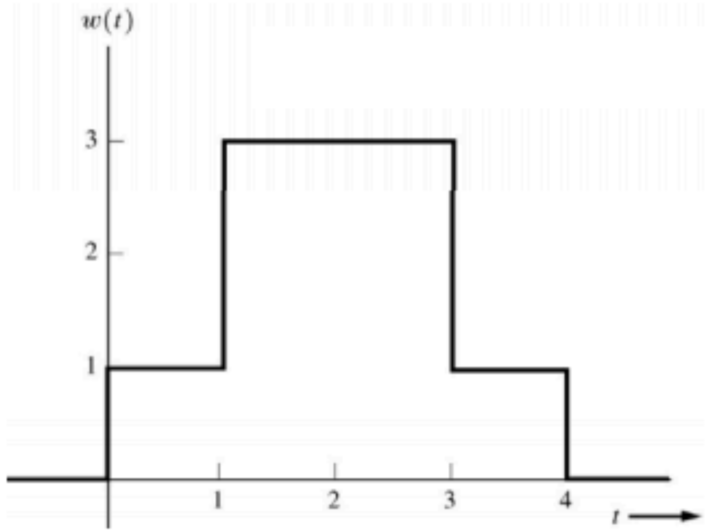
$$W(f) = 5\delta(f) - \frac{2.5}{1 + j\pi f}$$

```matlab
% Modify the Spectrum to include the Delta functions;
for (i = 1:1:length(f))
   if (f(i) == 0)  %only if f(i)=0
    Wmag(i) = 5;
    Theta(i)= 0;
   end;
end;
```



Spectrum of Rectangular Pulses

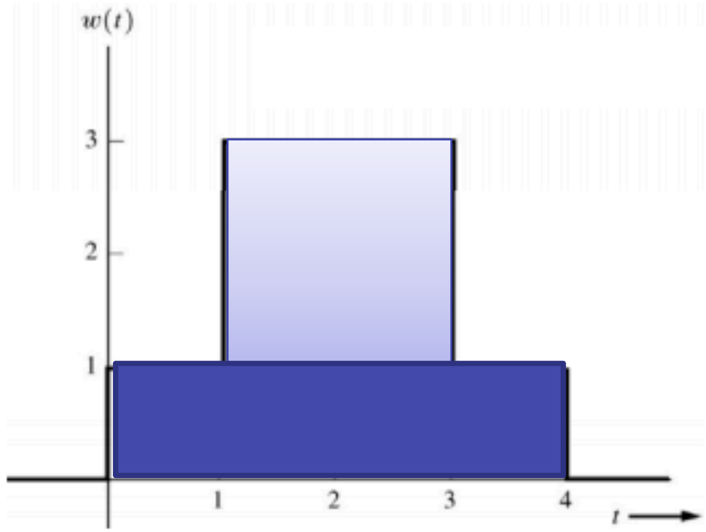
1. Find FT of w(t) waveform



What is w(t)?

Spectrum of Rectangular Pulses

1. Find FT of w(t) waveform



Solution: We use superposition of two rectangular pulses.

$$\omega(t) = \Pi\left(\frac{t-2}{4}\right) + 2\Pi\left(\frac{t-2}{2}\right)$$

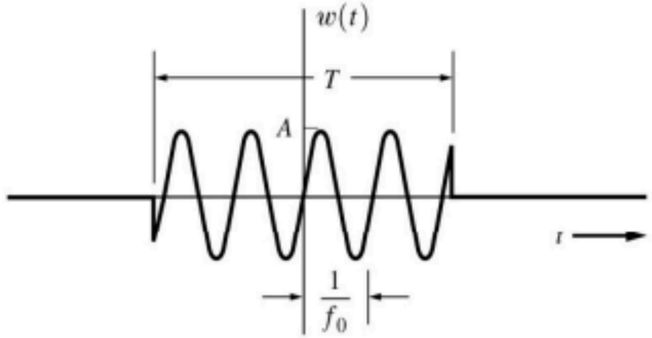
From FT tables, we find:

$$W(f) = 4\frac{\sin(4\pi f)}{4\pi f}e^{-j2\omega} + 2(2)\frac{\sin(2\pi f)}{2\pi f}e^{-j2\omega} = 4[\mathrm{Sa}(4\pi f) + Sa(2\pi f)]e^{-j4\pi f}$$

Spectrum of a Switched Sinusoid

Waveform of a switch sinusoid can be represented as follow:

$$\omega(t) = \Pi(\frac{t}{T}) A \sin \omega_0 t = \Pi(\frac{t}{T}) A \cos(\omega_0 t - \frac{\pi}{2})$$

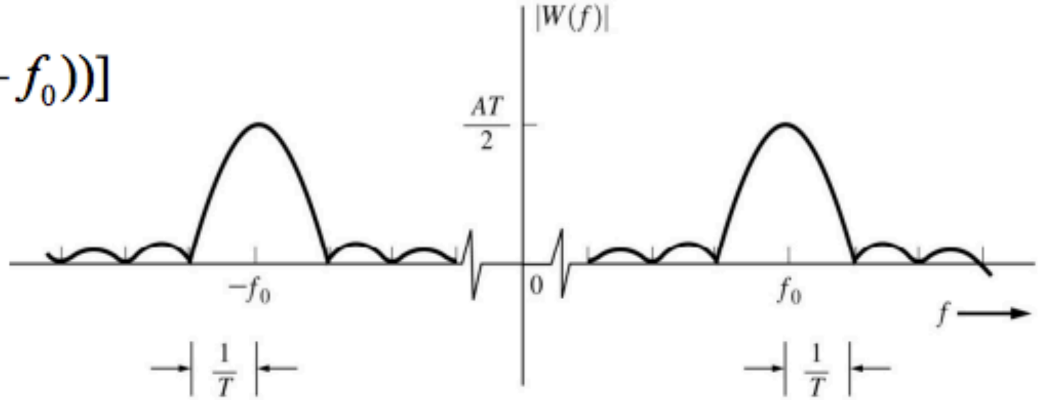


The frequency domain representation of w(t) will be:

$$W(f) = j\frac{A}{2}T[Sa(\pi T(f + f_0)) - Sa(\pi T(f - f_0))]$$

Note that the spectrum of w(t) is imaginary!

As T→INF, 1/T→ 0, then Sa waveform converges to a double-sided delta waveform



Magnitude Spectrum of w(t)

Alternative Tools

- Try the following:

  arg(1/(1+ix)); -100<x<100
  magnitude(1/(1+ix));

http://www.wolframalpha.com/input/?i=magnitude%281%2F%281%2Bix%29%29%3B+

- Another very interesting tool to demonstrate FT:

http://home.fuse.net/clymer/graphs/fourier.html

Try the following:

- sin(10*x)+sin(100*x)
- sin(10*x)+sin(100*x)
- exp(0.05*x)*sin(100*x)

Back to Properties of FT

- Spectral symmetry of real signals: If w(t) is real, w(t) = w*(t) then
  - $W(-f) = W^*(f)$, or $|W(f)|$ is even and $\theta(f)$ is odd.
  - $W(f)$ is real when $w(t)$ is even.
  - $W(f)$ is imaginary when $w(t)$ is odd.

- $|W(f)|^2 = \mathscr{E}(f)$ is called *Energy Spectral Density* in Joules/Hz &
- E = integral of $\mathscr{E}(f)$ w.r.t. freq.

- **Parseval's Theorem.**

$$\int_{-\infty}^{\infty} w_1(t) w_2^*(t)\,dt = \int_{-\infty}^{\infty} W_1(f) W_2^*(f)\,df$$

If w1(t)=w2(t)=w(t) →

- **Rayleigh's energy theorem,** which is

Energy Spectral Density! (Joules/Hz)

$$E = \int_{-\infty}^{\infty} |w(t)|^2 dt = \int_{-\infty}^{\infty} |W(f)|^2 df \rightarrow E = \int_{-\infty}^{\infty} \mathscr{E}(f)\,df$$

Total normalized energy

Power Spectral Density

- How the power content of signals and noise is distributed over different frequencies
- Useful in describing how the power content of signal with noise is affected by filters & other devices
- Important properties:
  - PSD is always a real nonnegative function of frequency
  - PSD is not sensitive to the phase spectrum of w(t) – due to absolute value operation
  - If the PSD is plotted in dB units, the plot of the PSD is identical to the plot of the Magnitude Spectrum in dB units
  - PSD has the unit of watts/Hz (or, equivalently, V^2/Hz or A^2/Hz)

Direct Method!

- PSD for a deterministic <u>power waveform</u> is

$$P_\omega(f) = \left(\lim_{T \to \infty} \frac{|W_T(f)|^2}{T} \right)$$

where $\omega_T(t) \leftrightarrow W_T(f)$ and $P_w(f)$ is in Watts/Hz.

- <u>Normalized average power:</u>

$$P = \langle \omega^2(t) \rangle = \int_{-\infty}^{\infty} P_\omega(f)\, df = W_{rms}^2$$

i.e., the area under PSD function.

Note that $|W(f)|^2$ was the <u>Energy Spectral Density</u> (ESD).

- $W_T(t)$ is the truncated version of the signal:

$$w_T(t) = \begin{cases} w(t), & -T/2 < t < T/2 \\ 0, & t \text{ elsewhere} \end{cases} = w(t)\Pi\left(\frac{t}{T}\right)$$

Any other way we can find PSD?→

Autocorrelation Function

Autocorrelation, *R(τ)*

- Relates power of a waveform to its freq.

$$R_\omega(\tau) = \langle \omega(t)\omega(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \omega(t)\omega(t+\tau)dt$$

- It can be shown that *PSD & autocorrelation function are FT pairs*. $R_\omega(\tau) \leftrightarrow P_\omega(f)$

  where $P_\omega(f) = F[R_\omega(\tau)]$

In summary PSD can be evaluated by either:

Direct Method!

$$P_\omega(f) = \left(\lim_{T\to\infty} \frac{|W_T(f)|^2}{T} \right)$$

Indirect Method! using FT of autocorrelation function:

$$P_\omega(f) = F[R_\omega(\tau)]$$

- Note that the total <u>average normalized power</u> of the waveform w(t) can be evaluated by any of the four techniques embedded in the formula below

$$P_{avg} = \langle \omega^2(t) \rangle = W_{rms}^2 = \int_{-\infty}^{\infty} P_\omega(f)\,df = R_\omega(0)$$

PSD

Example: Power Spectrum of a Sinusoid

Find the PSD of $w(t) = A \sin \omega_0 t$

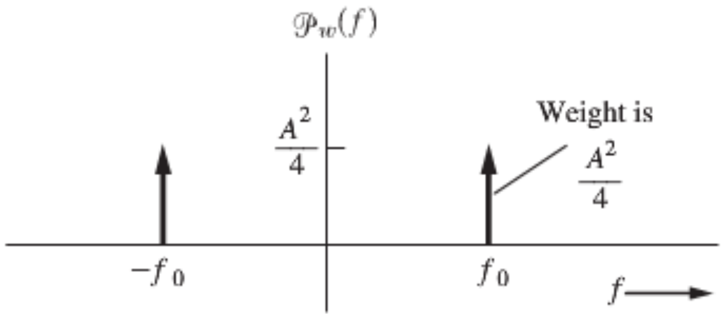
Method 2: using the indirect method (finding the autocorrelation): $P_\omega(f) = F[R_\omega(\tau)]$

$$R_w(\tau) = \langle w(t)w(t+\tau)\rangle$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A^2 \sin \omega_0 t \sin \omega_0 (t+\tau)\, dt$$

$$R_w(\tau) = \frac{A^2}{2}\cos \omega_0 \tau$$

$$\mathcal{P}_w(f) = \mathcal{F}\left[\frac{A^2}{2}\cos \omega_0 \tau\right] = \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$$



- Average normalized power

$$P = \int_{-\infty}^{\infty} \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]df = \frac{A^2}{2}$$

- We can verify this by

$$P = \langle \omega^2(t)\rangle = W_{rms}^2 = \left(A/\sqrt{2}\right)^2 = \frac{A^2}{2}$$

1. $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
2. $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$
3. $\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$
4. $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
5. $\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$
6. $\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$
7. $\sin 2a = 2 \sin a \cos a$
8. $\cos^2 a = \frac{1}{2}(1 + \cos 2a)$
9. $\sin^2 a = \frac{1}{2}(1 - \cos 2a)$

Orthogonal & Orthonormal Functions

- Orthogonal Function: $\varphi_n(t)$ and $\varphi_m(t)$ are orthogonal if

$$\int_a^b \varphi_n(t)\varphi_m^*(t)\,dt = 0 \quad \text{for } n \neq m$$

Over some interval a & b

Orthogonal functions are independent, in disagreement, unlikely!

- Orthonormal Function: $\varphi_n(t)$ and $\varphi_m(t)$ are orthonormal if

$$\int_a^b \varphi_n(t)\varphi_m^*(t)\,dt = \begin{cases} 0, & n \neq m \\ K_n, & n = m \end{cases} = K_n \delta_{nm}$$

$$\delta_{nm} \triangleq \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$$

& Kn = 1

Note that if Kn is any constant other than unity, then the functions are not orthonormal!

Example

- Show that $\varphi_1(t)=\Pi(t)$ *and* $\varphi_2(t)=\sin 2\pi t$ are orthogonal functions over the interval $-0.5<t<0.5$.

<u>Solution:</u> $$\int_a^b \varphi_1(t)\varphi_2(t)dt = \int_{-0.5}^{0.5} 1\sin 2\pi t dt = -\frac{\cos 2\pi t}{2\pi}\Bigg|_{-0.5}^{0.5} = \frac{-1}{2\pi}\left[\cos\pi - \cos(-\pi)\right] = 0$$

Seems like two functions are always orthogonal!!!!

Note that $\Pi(t)$ and $\sin 2\pi t$ are not orthogonal over the interval $0<t<1$ because $\Pi(t)=0$ *for* $t>0.5$, & the integral from 0 to 0.5 is $1/\pi$ which is not zero.

Can you show this?

Orthogonal Series

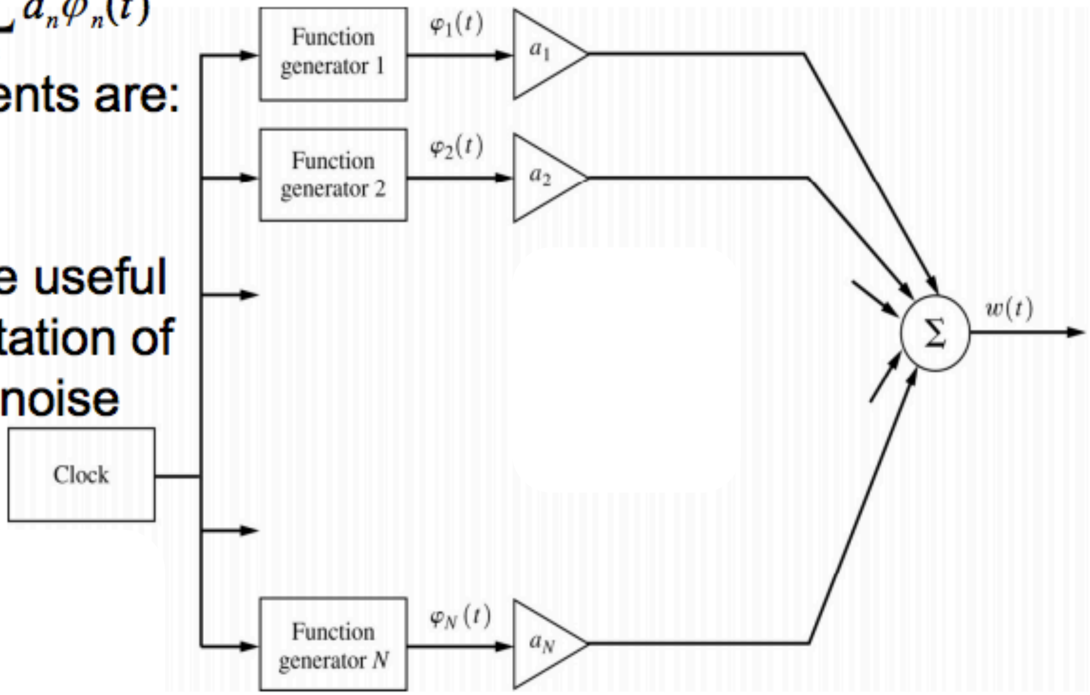
- A physical waveform *w(t)* that comprises signal + noise can be represented over interval *a<t<b* by a series: $\omega(t)=\sum_n a_n\varphi_n(t)$

  & the orthogonal coefficients are:

$$a_n=\frac{1}{K_n}\int_a^b \omega(t)\varphi_n^*(t)\,dt$$

- The orthogonal series are useful in deterministic representation of signal, noise, and signal-noise combination.

Note that a_n and $\phi_n(t)$ are orthogonal



w(t) → orthogonal series → $\omega(t)=\sum_n a_n\varphi_n(t)$

Example

- Are sets of complex exponential functions ($e^{jn\omega_0 t}$) over the interval a<t<b=a+To, $w_o=2\pi/To$ orthogonal? Are they orthonormal?

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$$\int_a^b \varphi_n(t)\varphi_m^*(t)dt = \int_a^{a+T_0} e^{jn\omega_0 t}e^{-jm\omega_0 t}dt = \int_a^{a+T_0} e^{j(n-m)\omega_0 t}dt$$

For n ≠ m, we have $\int_a^{a+T_0} e^{j(n-m)\omega_0 t}dt = \frac{e^{j(n-m)\omega_0 a}[e^{j(n-m)2\pi} - 1]}{j(n-m)\omega_0} = 0$

since $e^{j(n-m)2\pi} = \cos[2\pi(n-m)] + j\sin[2\pi(n-m)] = 1$, the orthogonality is satisfied.

- For n=m, $\int_a^{a+T_0} \varphi_n(t)\varphi_m^*(t)dt = \int_a^{a+T_0} 1dt = T_0$

$K_n = T_0$ and because $K_0 \neq 1$, the function is not orthonormal but orthogonal.

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- M. Farooque Mesiya, Contemporary Communication Systems, 2012 – Chapter 2
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  (https://www.goodreads.com/book/show/209442.Electronic_Communications_System)