4. ELECTROSTATICS

Remember…

Electric Energy can be propagated by EM waves
EM waves are created by Oscillating E and H
The faster the Oscillation is more propagation we have (radio waves)

Table 1-3: The three branches of electromagnetics.

Branch	Condition	Field Quantities (Units)
Electrostatics	Stationary charges ($\partial q/\partial t = 0$)	Electric field intensity **E** (V/m) Electric flux density **D** (C/m^2) $\mathbf{D} = \varepsilon \mathbf{E}$
Magnetostatics	Steady currents ($\partial I/\partial t = 0$)	Magnetic flux density **B** (T) Magnetic field intensity **H** (A/m) $\mathbf{B} = \mu \mathbf{H}$
Dynamics (Time-varying fields)	Time-varying currents ($\partial I/\partial t \neq 0$)	**E**, **D**, **B**, and **H** (**E**, **D**) coupled to (**B**, **H**)

Basic Idea

Flux Density (1/m^2) $\leftarrow$ Field Intensity (1/m)

$D \leftarrow E$

$B \leftarrow H$

EM Waves can be
Static or Time Varying

Electric Charges q $\rightarrow$ Fe

Eclectic Current I $\rightarrow$ V

Current Density J $\rightarrow$ I

Maxwell's Equations

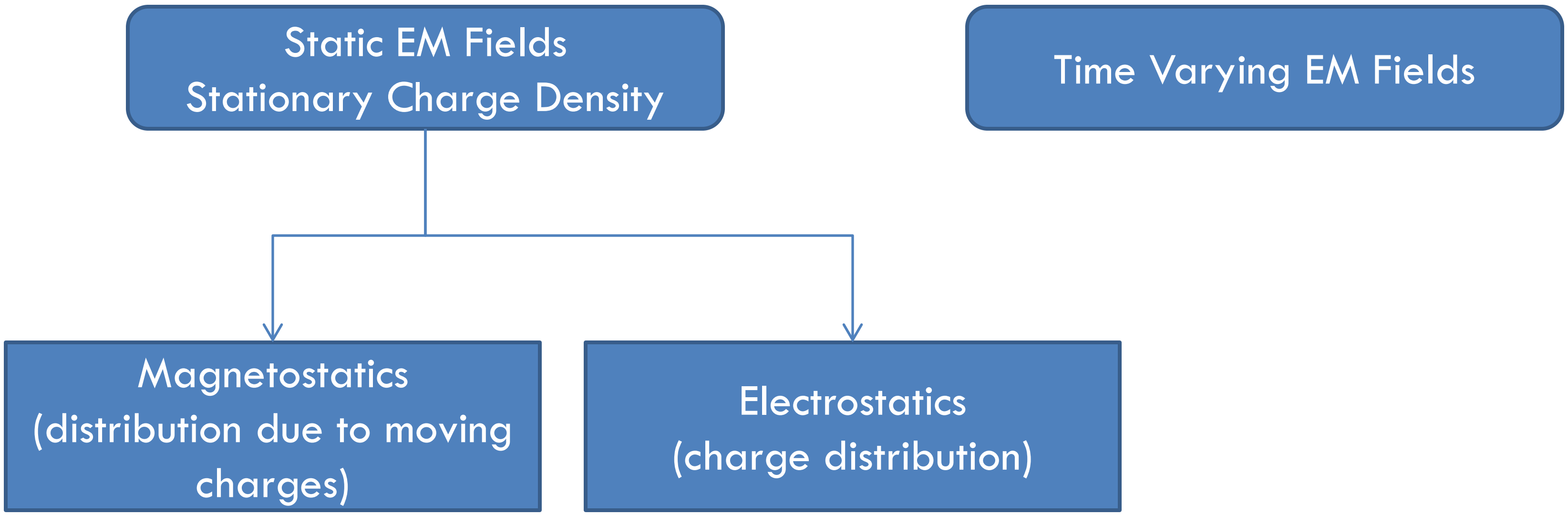
Static EM Fields
Stationary Charge Density

Time Varying EM Fields

Fixed in Space
Steady State Rate (no rate of change in time)

EM Fields

Fixed in Space
Steady State Rate (no rate of change in time)



Charge Distributions

Volume charge density:

$$\rho_v = \lim_{\Delta \mathcal{V} \to 0} \frac{\Delta q}{\Delta \mathcal{V}} = \frac{dq}{d\mathcal{V}} \qquad (\text{C/m}^3)$$

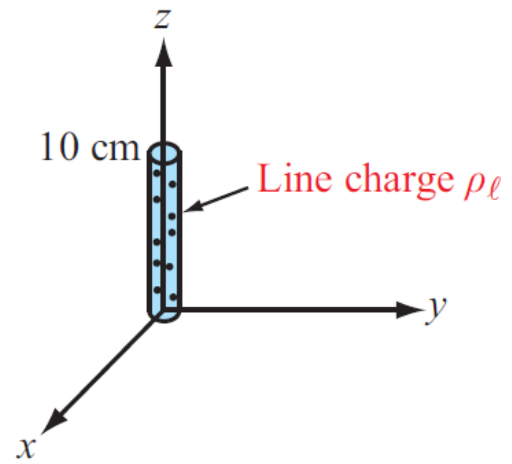
Total Charge in a Volume

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} \qquad (\text{C})$$

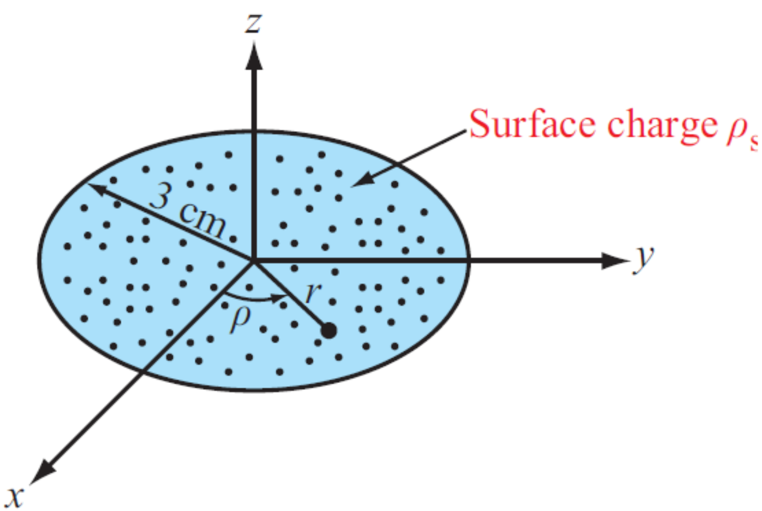
Surface and Line Charge Densities

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \qquad (\text{C/m}^2)$$

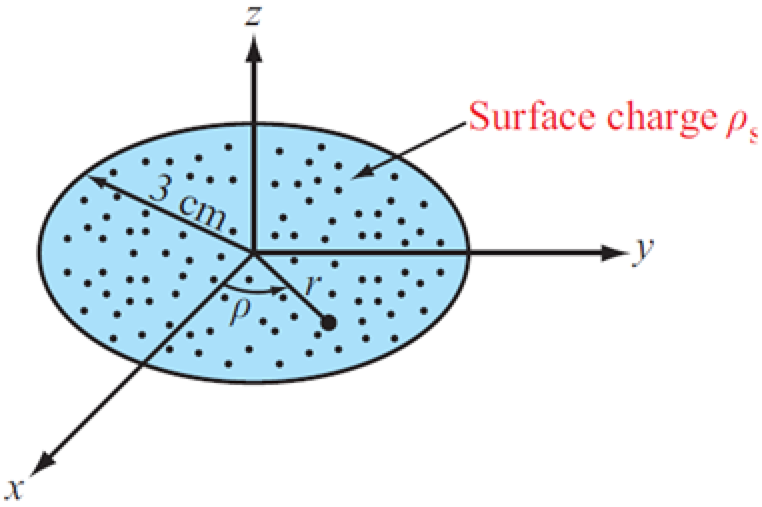
$$\rho_\ell = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \qquad (\text{C/m})$$



(a) Line charge distribution



Example (A)



Current Density

If a volume of charge density moves → Current density will be generated

C/m^3 x m/s → A/m^2

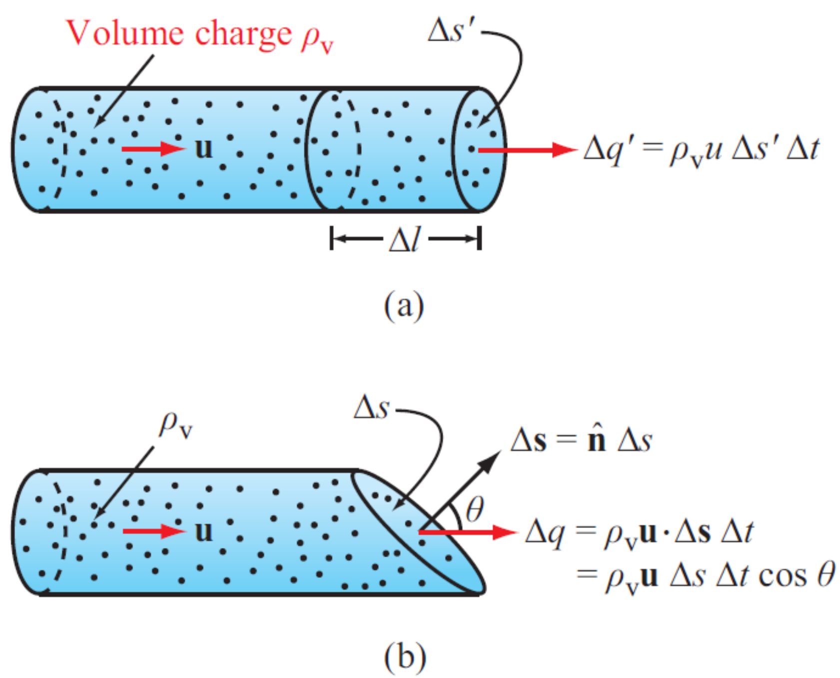


Figure 4-2: Charges with velocity $\mathbf{u}$ moving through a cross section $\Delta s'$ in (a) and Δs in (b).

$$\mathbf{J} = \rho_v \mathbf{u} \qquad (\text{A/m}^2) \qquad (4.11)$$

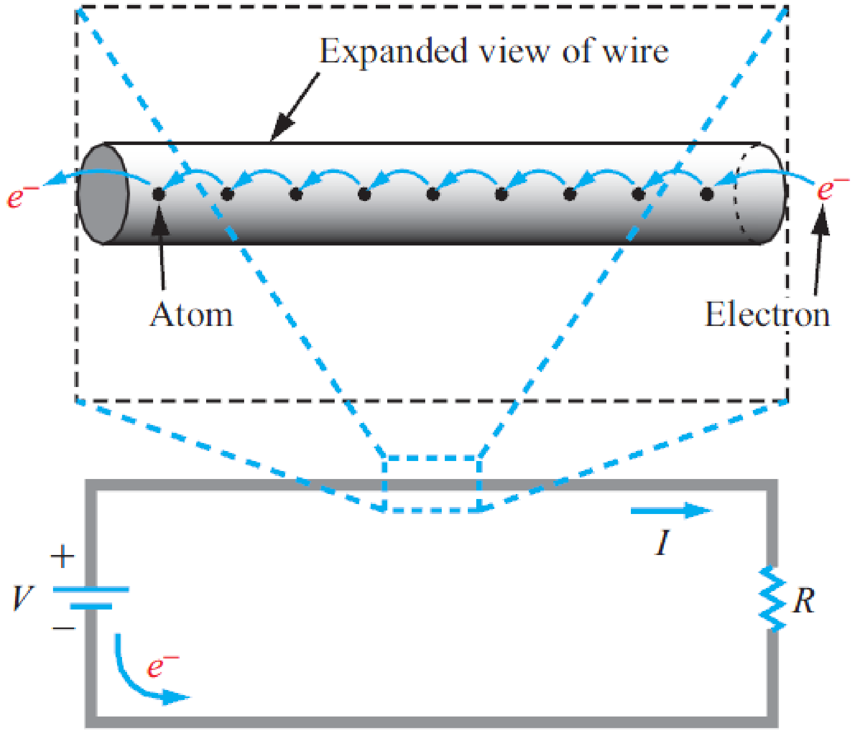
$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad (\text{A}). \qquad (4.12)$$

J is called the current density

*When a current is due to the actual movement of electrically charged matter, it is called a **convection current**, and **J** is called a **convection current density**.*

Convection vs. Conduction

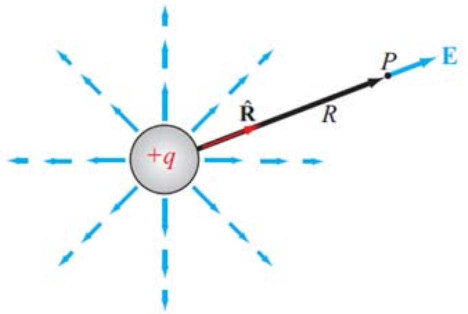
When a current is due to the movement of charged particles relative to their host material, **J** is called a ***conduction current density***.

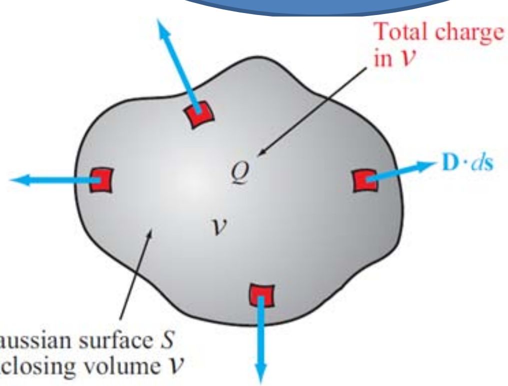


Two Important Laws

Coulomb's Law
(Relation ship between charges and E-Field Expression)

Gauss's Law
(Relationship between charges and Electric Flux Density D)





Electric Field

Single Charge

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \qquad \text{(V/m)}$$

$$\mathbf{F} = q'\mathbf{E} \qquad \text{(N)}$$

Charge Distribution

Multiple Charges

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i(\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3} \qquad \text{(V/m)}.$$

$$\mathbf{E} = \int_{\mathcal{V}'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \hat{\mathbf{R}}' \frac{\rho_{\mathrm{v}}\, d\mathcal{V}'}{R'^2}$$

(volume distribution). (4.21a)

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_{\mathrm{s}}\, ds'}{R'^2} \qquad \text{(surface distribution)},$$

(4.21b)

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_{\ell}\, dl'}{R'^2} \qquad \text{(line distribution)}.$$

(4.21c)

Coulomb's Law

Electric field at point P due to single charge

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \qquad \text{(V/m)}$$

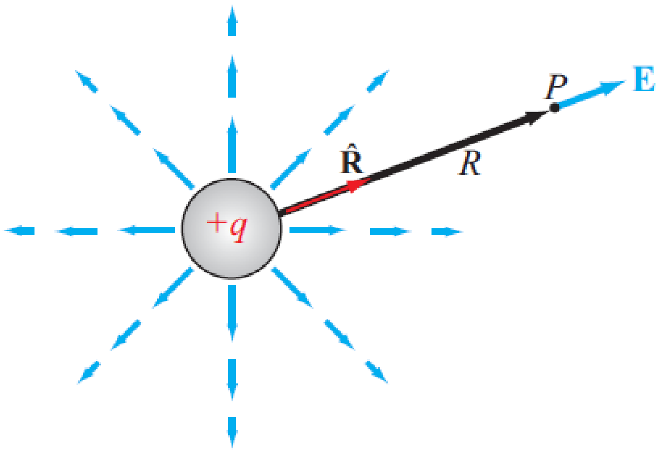


Figure 4-3: Electric-field lines due to a charge q.

Electric force on a test charge placed at *P*

$$\mathbf{F} = q'\mathbf{E} \qquad \text{(N)}$$

Electric flux density **D**

$$\mathbf{D} = \varepsilon \mathbf{E}$$

If ε is independent of the magnitude of **E**, *then the material is said to be* ***linear*** *because* **D** *and* **E** *are related linearly, and if it is independent of the direction of* **E**, *the material is said to be* ***isotropic.***

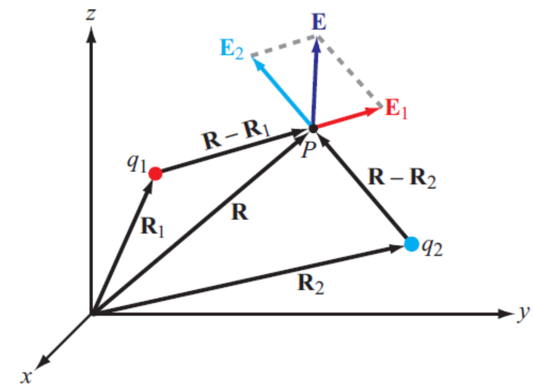
$$\varepsilon = \varepsilon_r \varepsilon_0,$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \simeq (1/36\pi) \times 10^{-9} \qquad \text{(F/m)}$$

Electric Field due to Multiple Charges (Example B)

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i(\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3} \quad \text{(V/m)}.$$

Two point charges with $q_1 = 2 \times 10^{-5}$ C and $q_2 = -4 \times 10^{-5}$ C are located in free space at points with Cartesian coordinates $(1, 3, -1)$ and $(-3, 1, -2)$, respectively. Find (a) the electric field $\mathbf{E}$ at $(3, 1, -2)$ and (b) the force on a 8×10^{-5} C charge located at that point. All distances are in meters.



Find R1, R2, R-R1, R-R2

Examples C & D

Electric Field Due to Charge Distributions

Field due to:

a differential amount of charge $dq = \rho_v \, d\mathcal{V}'$ contained in a differential volume $d\mathcal{V}'$ is

$$d\mathbf{E} = \hat{\mathbf{R}}' \frac{dq}{4\pi\varepsilon R'^2} = \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{4\pi\varepsilon R'^2}, \qquad (4.20)$$

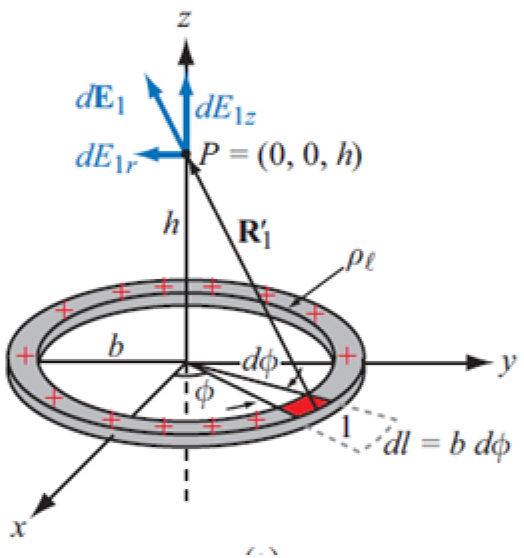
$$\mathbf{E} = \int_{\mathcal{V}'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{R'^2}$$

(volume distribution). (4.21a)

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2} \quad \text{(surface distribution)}, \qquad (4.21b)$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2} \quad \text{(line distribution)}. \qquad (4.21c)$$

Unit Vector in R direction!



Electric Field Due to surface Distribution
(a piece of wire)

Example

A ring of charge of radius b is characterized by a uniform line charge density of positive polarity ρ_ℓ. The ring resides in free space and is positioned in the x–y plane

Determine the electric field intensity $\mathbf{E}$ at a point $P = (0, 0, h)$ along the axis of the ring at a distance h from its center.

$$Q = 2\pi b \rho_\ell$$

Segment length $dl = b\,d\phi$

$$dq = \rho_\ell\, dl = \rho_\ell b\, d\phi.$$

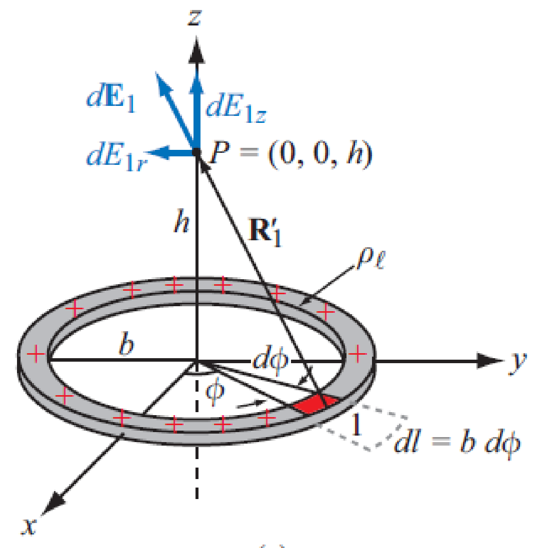
$$\mathbf{R}'_1 = -\hat{\mathbf{r}}b + \hat{\mathbf{z}}h,$$

$$R'_1 = |\mathbf{R}'_1| = \sqrt{b^2 + h^2}\,, \qquad \hat{\mathbf{R}}'_1 = \frac{\mathbf{R}'_1}{|\mathbf{R}'_1|} = \frac{-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h}{\sqrt{b^2 + h^2}}\,.$$

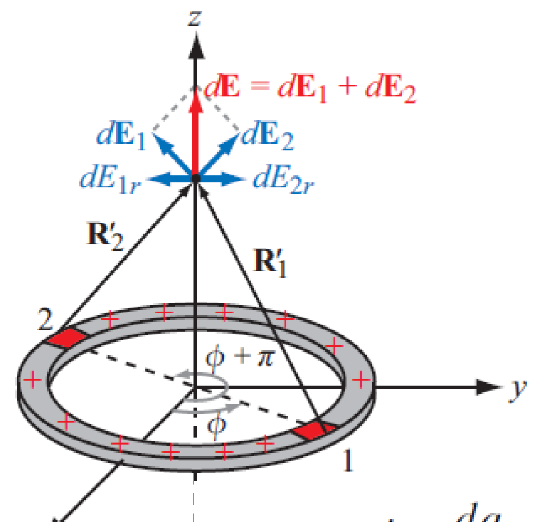
$$d\mathbf{E}_1 = \frac{1}{4\pi\varepsilon_0}\,\hat{\mathbf{R}}'_1\,\frac{\rho_\ell\, dl}{R_1'^2} = \frac{\rho_\ell b}{4\pi\varepsilon_0}\,\frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2 + h^2)^{3/2}}\,d\phi.$$

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{\mathbf{z}}\,\frac{\rho_\ell b h}{2\pi\varepsilon_0}\,\frac{d\phi}{(b^2 + h^2)^{3/2}}\,.$$

$$\mathbf{E} = \hat{\mathbf{z}}\,\frac{\rho_\ell b h}{2\pi\varepsilon_0 (b^2 + h^2)^{3/2}} \int_0^{\pi} d\phi = \hat{\mathbf{z}}\,\frac{\rho_\ell b h}{2\varepsilon_0 (b^2 + h^2)^{3/2}}$$



(a)



$$d\mathbf{E} = \hat{\mathbf{R}}'\,\frac{dq}{4\pi\varepsilon R'^2}$$

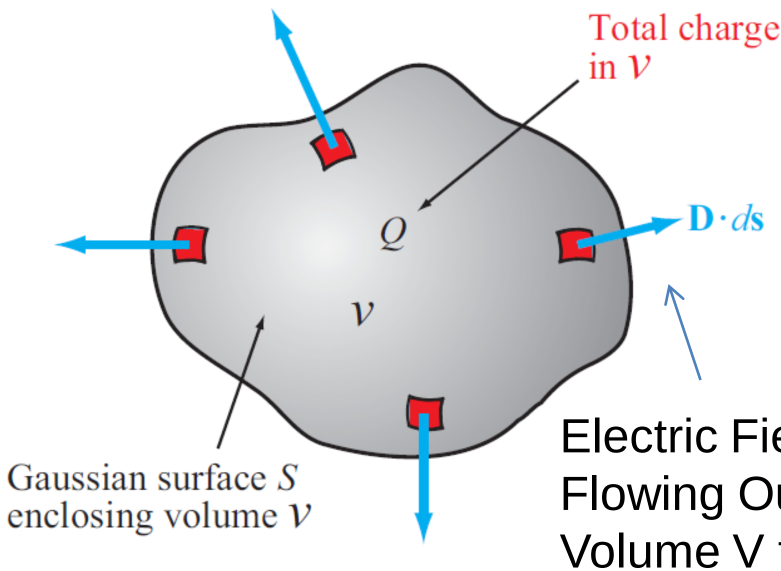
$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_{l'} \hat{\mathbf{R}}'\,\frac{\rho_\ell\, dl'}{R'^2} \quad \text{(line distribution).}$$

Gauss's Law

$$\nabla \cdot \mathbf{D} = \rho_v$$

(Differential form of Gauss's law),

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} = Q$$



Electric Field Flux Flowing Outward of Volume V through ds

Application of the divergence theorem gives:

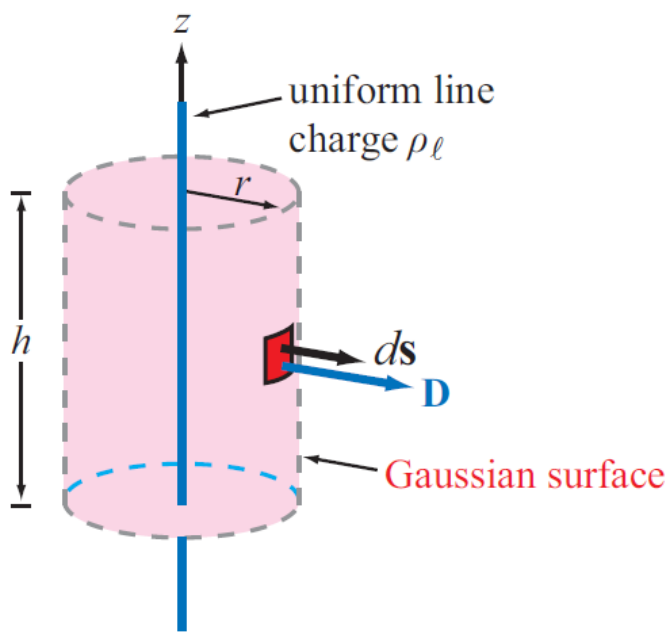
$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \oint_S \mathbf{D} \cdot d\mathbf{s}. \qquad (4.28)$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.29)$$

(Integral form of Gauss's law).

Total Flux through surface S = Total Charges;
We call S is the Gaussian Surface

Applying Gauss's Law
Example E



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

Use Gauss's law to obtain an expression for **E** due to an infinitely long line with uniform charge density ρ_ℓ that resides along the z-axis in free space.

Construct an imaginary Gaussian cylinder of radius *r* and height *h*:

Line Charge Density

$$\text{Total Charge} = \int_{z=0}^{h} \int_{\phi=0}^{2\pi} \hat{\mathbf{r}} D_r \cdot \hat{\mathbf{r}} r \, d\phi \, dz = \rho_\ell h$$

or

$$2\pi h D_r r = \rho_\ell h,$$

which yields

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \frac{\rho_\ell}{2\pi \varepsilon_0 r} \quad (4.33)$$

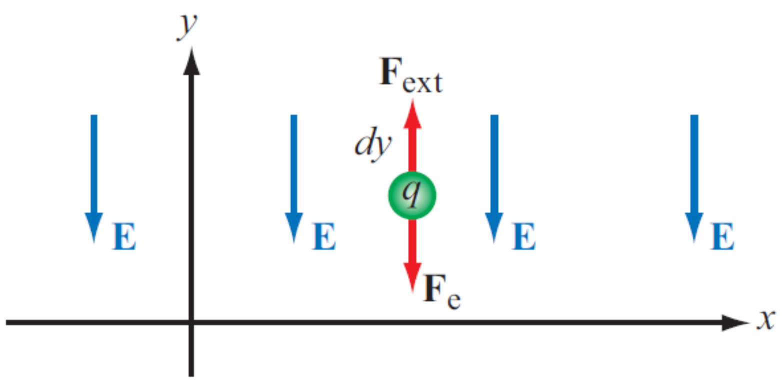
(infinite line charge).

Example F

Example G

Electric Scalar Potential

The term "voltage" is short for "voltage potential" and synonymous with ***electric potential****.*



Minimum force needed to move charge against **E** field:

$$\mathbf{F}_{ext} = -\mathbf{F}_e = -q\mathbf{E}.$$

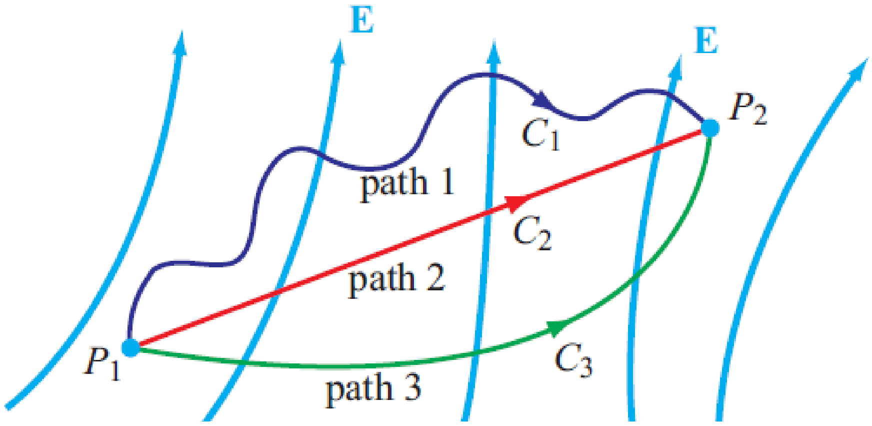
$$dW = \mathbf{F}_{ext} \cdot d\mathbf{l} = -q\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J)}.$$

$$dW = -q(-\hat{\mathbf{y}}E) \cdot \hat{\mathbf{y}}\, dy = qE\, dy.$$

Differential Electric Potential:

$$dV = \frac{dW}{q} = -\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J/C or V)}.$$

Electric Scalar Potential



$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l},$$

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \qquad (4.39)$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{(Electrostatics).} \qquad (4.40)$$

For point Charge and continuous charge distributions:

$$V = -\int_{\infty}^{R} \left(\hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \right) \cdot \hat{\mathbf{R}} \, dR = \frac{q}{4\pi\varepsilon R} \qquad \text{(V)}. \qquad (4.45)$$

Note:

$$V = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{R} - \mathbf{R}_i|} \qquad \text{V}$$

$$V = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_v}{R'} \, d\mathcal{V}' \quad \text{(volume distribution)}, \qquad (4.48a)$$

$$V = \frac{1}{4\pi\varepsilon} \int_{S'} \frac{\rho_s}{R'} \, ds' \quad \text{(surface distribution)}, \qquad (4.48b)$$

$$V = \frac{1}{4\pi\varepsilon} \int_{l'} \frac{\rho_\ell}{R'} \, dl' \quad \text{(line distribution)}. \qquad (4.48c)$$

Example (K)

Relating **E** to *V*

$$dV = -\mathbf{E} \cdot d\mathbf{l}.$$

By Definition

$$dV = \nabla V \cdot d\mathbf{l},$$

Thus:

$$\mathbf{E} = -\nabla V.$$

This is the differential relationship between E and V

Poisson's & Laplace's Equations

$$\nabla \cdot \mathbf{D} = \rho_v$$

(Differential form of Gauss's law),

$$\mathbf{D} = \varepsilon \mathbf{E}, \longrightarrow \nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} .$$

We know $\mathbf{E} = -\nabla V.$

$$\nabla \cdot (\nabla V) = -\frac{\rho_v}{\varepsilon} .$$

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} ,$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \text{(Poisson's equation).}$$

For Example:

$$V = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho_v}{R'} \, dV'$$

In the absence of charges:

$$\nabla^2 V = 0 \qquad \text{(Laplace's equation),}$$

Conduction Current

Constitutive Parameters: Permeability, Permittivity, Conductivity
Homogeneous Materials: Constitutive Parameters are the same for all the points
Isotropic Materials: Constitutive Parameters will not change due to field direction

Conduction current density:

$$\mathbf{J} = \sigma \mathbf{E} \qquad (\text{A/m}^2) \quad (\text{Ohm's law}),$$

Materials: Conductors & Dielectrics
Conductors: Loose electrons → Conduction current can be created due to E field
Dielectrics: electrons are tightly bound to the atom → no current when E is applied

Conductors	
Silver	6.2×10^7
Copper	5.8×10^7
Gold	4.1×10^7
Aluminum	3.5×10^7
Iron	10^7
Mercury	10^6
Carbon	3×10^4
Semiconductors	
Pure germanium	2.2
Pure silicon	4.4×10^{-4}
Insulators	
Glass	10^{-12}
Paraffin	10^{-15}
Mica	10^{-15}
Fused quartz	10^{-17}

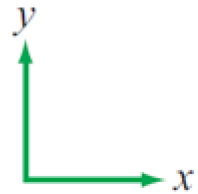
Perfect dielectric: $\mathbf{J} = 0$,

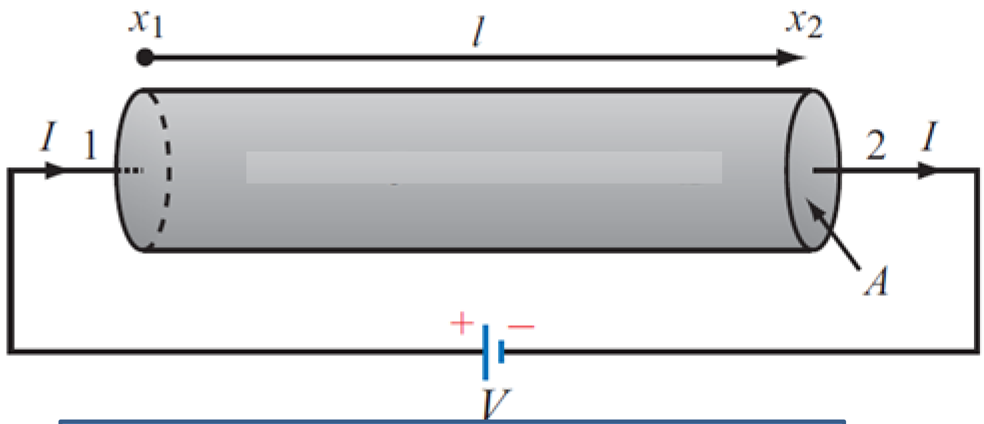
Perfect conductor: $\mathbf{E} = 0$.

Conductivity depends on impurity and temperature!

For metals: T inversely proportional to Conductivity!

Resistance (for a cylindrical wire)





What are I, E, J directions?

Longitudinal Resistor

$$V = V_1 - V_2 = -\int_{x_2}^{x_1} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{x_2}^{x_1} \hat{\mathbf{x}} E_x \cdot \hat{\mathbf{x}}\, dl = E_x l \qquad \text{(V)}.$$

$$I = \int_A \mathbf{J} \cdot d\mathbf{s} = \int_A \sigma \mathbf{E} \cdot d\mathbf{s} = \sigma E_x A \qquad \text{(A)}.$$

For any conductor:

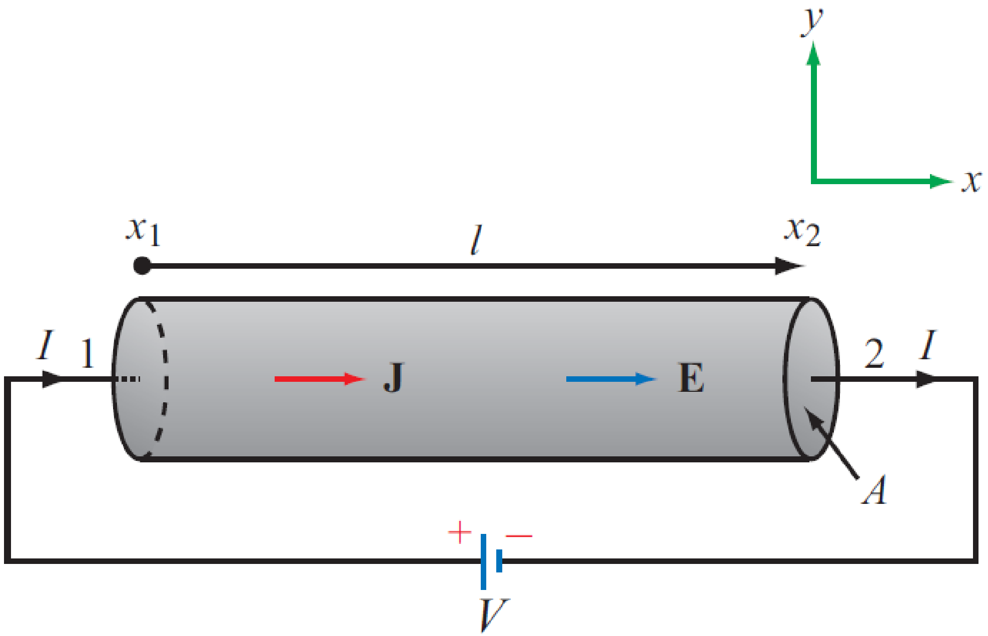
$$R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}\,.$$

R=V/I (the above equations)

$$R = \frac{l}{\sigma A} \qquad (\Omega).$$

For Cylindrical Wire

Resistance



Longitudinal Resistor

$$V = V_1 - V_2 = -\int_{x_2}^{x_1} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{x_2}^{x_1} \hat{\mathbf{x}} E_x \cdot \hat{\mathbf{x}}\, dl = E_x l \qquad \text{(V)}.$$

For any conductor:

$$R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}.$$

$$I = \int_A \mathbf{J} \cdot d\mathbf{s} = \int_A \sigma \mathbf{E} \cdot d\mathbf{s} = \sigma E_x A \qquad \text{(A)}.$$

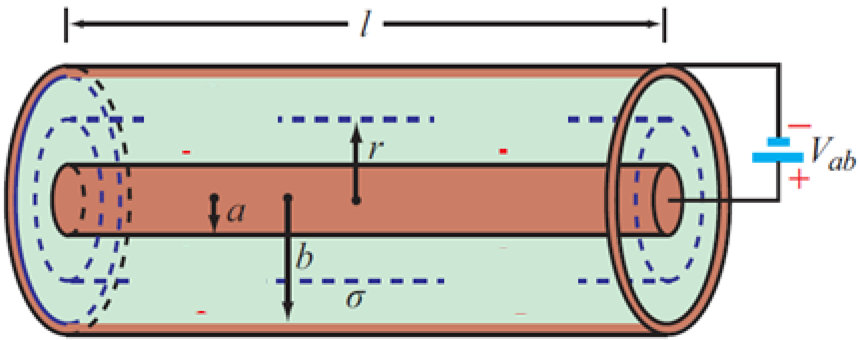
R=V/I (the above equations)

$$R = \frac{l}{\sigma A} \qquad (\Omega).$$

Used for sensors to measure pressure

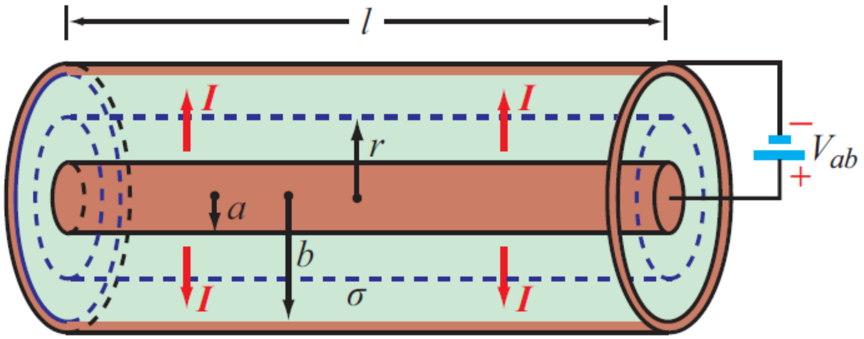
Example

- Find directions of Current, E, I , J:



Example (H)

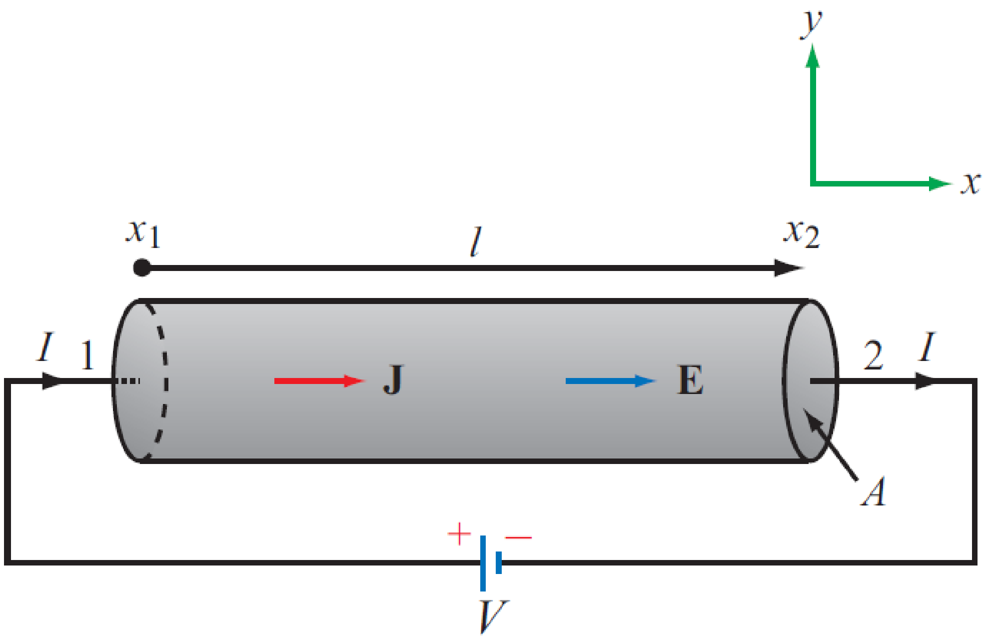
- Find directions of Current, E, I , J:



Given I and assuming perfect conductor;
Find E, Vab, and R & G for the dielectric and the dissipated power in the coax!

Example (J)

- Assume conductivity of copper is 5.8 x 10^7 (S/m)
- Assume V=1.5 mV, r=2 cm; l=50 m
- Find R and P of the cop

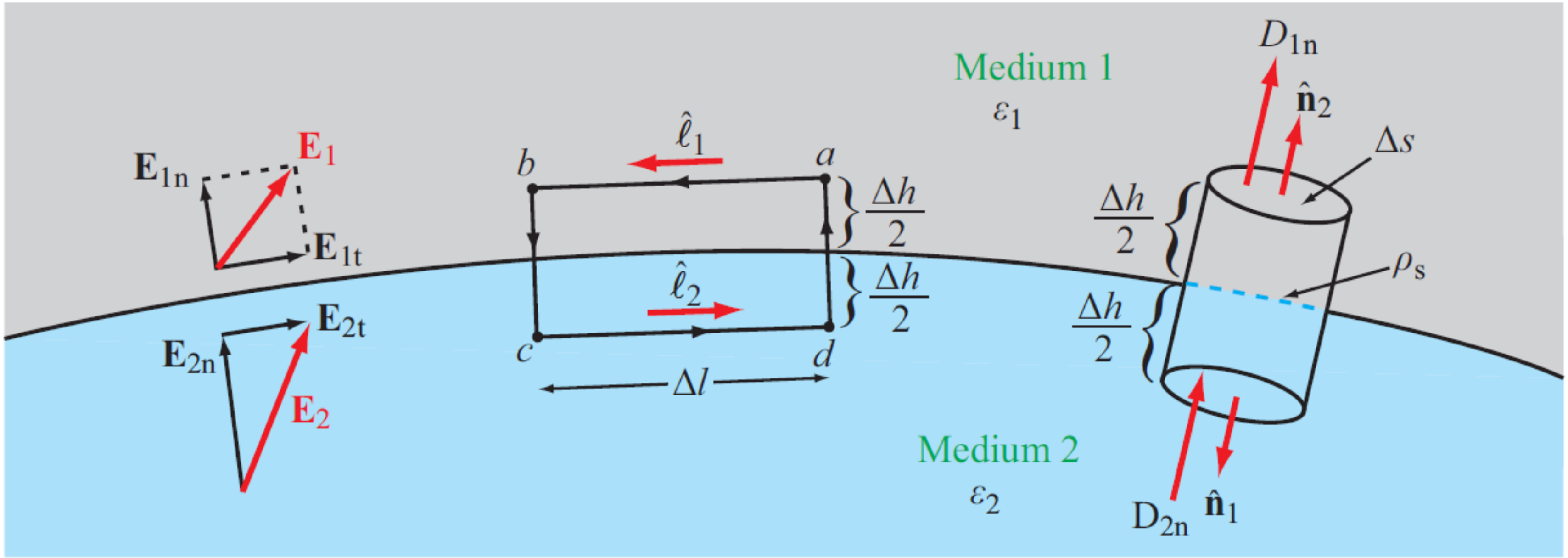


HW

- Do the suggested problems
- Create a table

Boundary Conditions

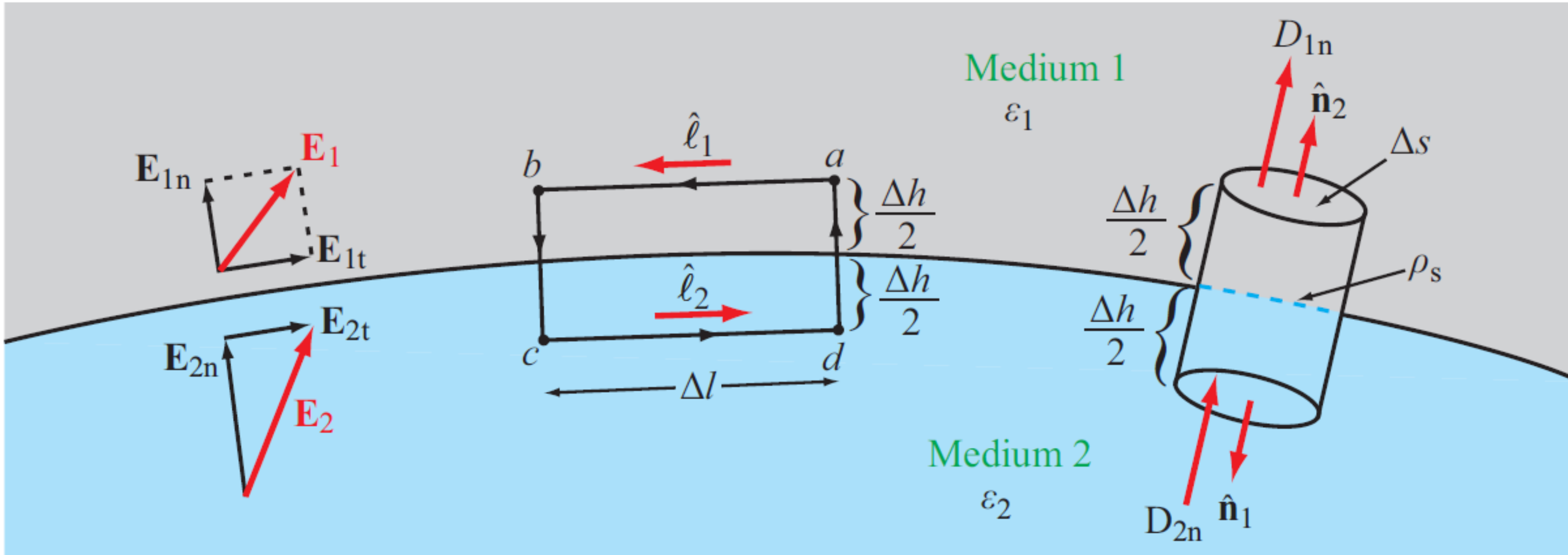
n1 and n2 are unit vectors directed normally outward



l1 and l2 are unit vectors directed along the tangent lines

Boundary Conditions

n1 and n2 are unit vectors directed normally outward



How do fields change at boundaries?

$$\mathbf{E}_{1t} = \mathbf{E}_{2t} \qquad \text{(V/m)}.$$

$$\frac{\mathbf{D}_{1t}}{\varepsilon_1} = \frac{\mathbf{D}_{2t}}{\varepsilon_2}.$$

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \qquad (\text{C/m}^2).$$

$$D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2).$$

Normal component of D changes abruptly when there is a charged boundary in an amount equal surface charge density!

Summary of Boundary Conditions

Two Dielectrics

Field Component	Any Two Media
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$
Tangential D	$\mathbf{D}_{1t}/\varepsilon_1 = \mathbf{D}_{2t}/\varepsilon_2$
Normal E	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$
Normal D	$D_{1n} - D_{2n} = \rho_s$

Remember $\mathbf{E} = 0$ in a good conductor

Summary of Boundary Conditions

Dielectric and Conductor (with E=0) / Note: J=σE

Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\varepsilon_1 = \mathbf{D}_{2t}/\varepsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

$\rho s = \varepsilon E$

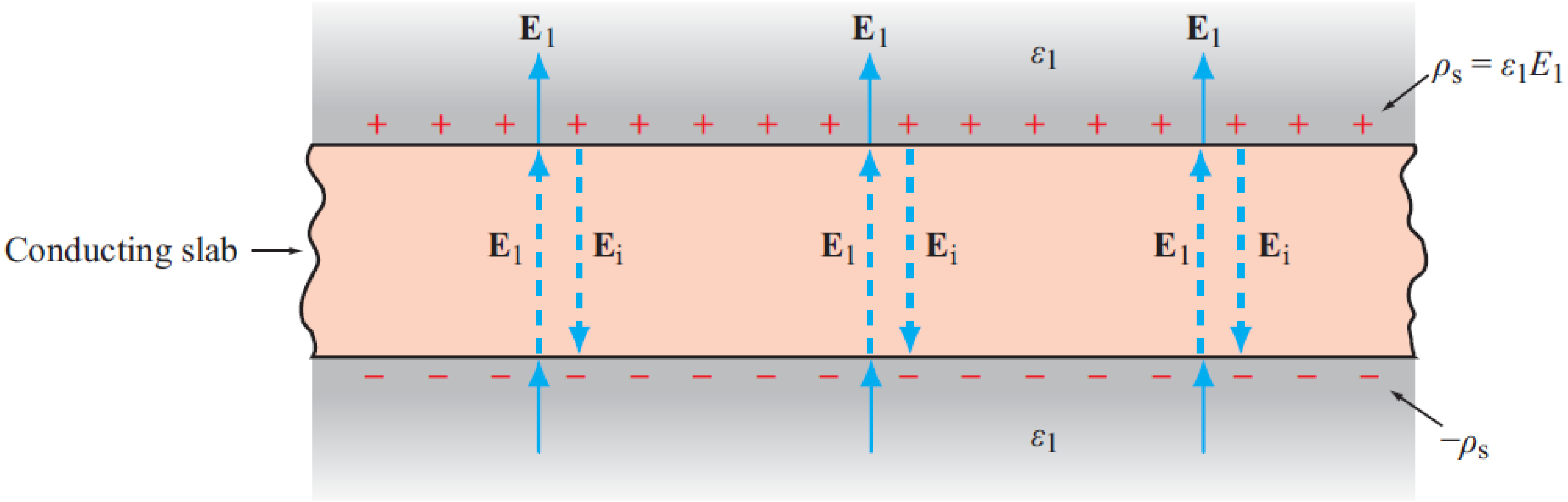
Remember **E** = 0 in a good conductor

Conductors

Net electric field inside a conductor is zero

E1 = -Ei

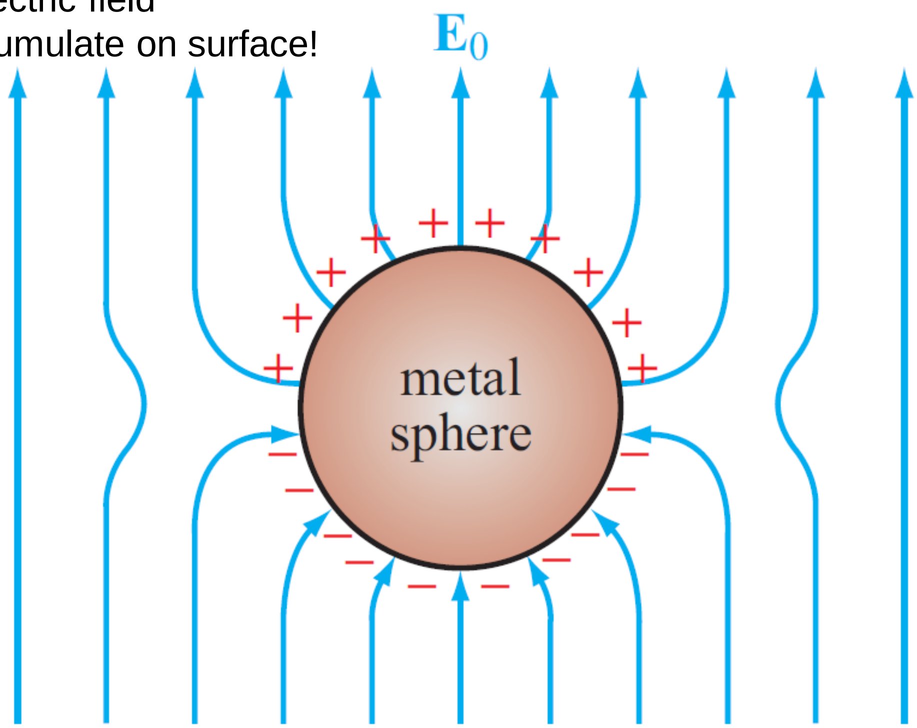
Induces surface charg



Note that E is always normal to a conductor boundary!

Field Lines at Conductor Boundary

We place a sphere in an electric field
→ + and – charges will accumulate on surface!



Metal sphere placed in an external electric field $\mathbf{E}_0$.

At conductor boundary, **E** field direction is always perpendicular to conductor surface

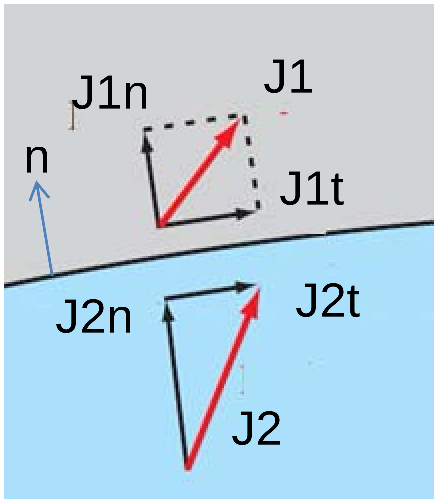
Summary of Boundary Conditions

Conductor and Conductor (E=0) / Note: $J=\sigma E$

Field Component	Any Two Media	Two conductors
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$ $J_{1t}/\sigma_1 = J_{2t}/\sigma_2$
Tangential D	$\mathbf{D}_{1t}/\varepsilon_1 = \mathbf{D}_{2t}/\varepsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t}$
Normal E	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$	$J_{1n}(\varepsilon_1/\sigma_1) - J_{2n}(\varepsilon_2/\sigma_2) = \rho_s$ But $J_{1n} = J_{2n}$
Normal D	$D_{1n} - D_{2n} = \rho_s$	

Note that if J_{1n} not equal J_{2n} →
Amount of charges arriving and leaving the boundary will be different → ρ_s with change over time (not true for Electrostatics)
→ $J_{1n} = J_{2n}$

Summary of Boundary Conditions



Normal E

Normal D

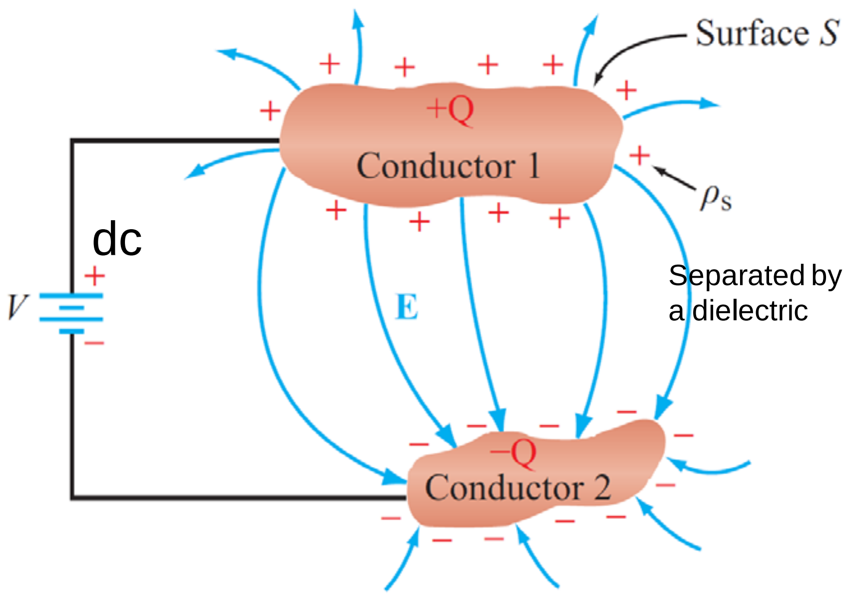
Amount of charges a

Under electrostatic conditions, normal components of J has to be continuous across the boundary between two different media

Examples (L)

Capacitance

Conductor has access charges → charges will be accumulated on the surface → E=0 everywhere within the conductor →V will be the same at every point in the conductor!



The ***capacitance*** of a two-conductor configuration is defined as

$$C = \frac{Q}{V} \qquad \text{(C/V or F)},$$

Potential diff between conductors

Capacitance

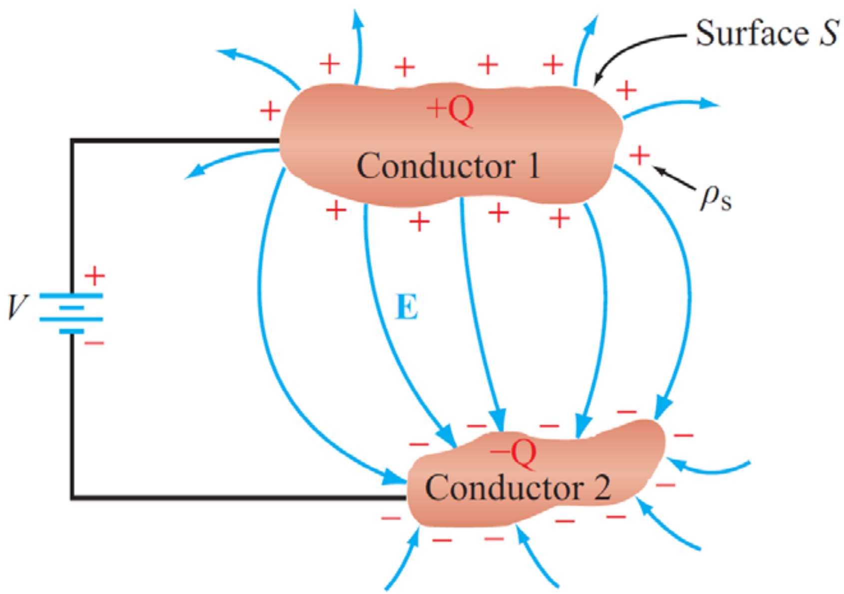
For any two-conductor configuration:

$$C = \frac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}} \qquad \text{(F)},$$

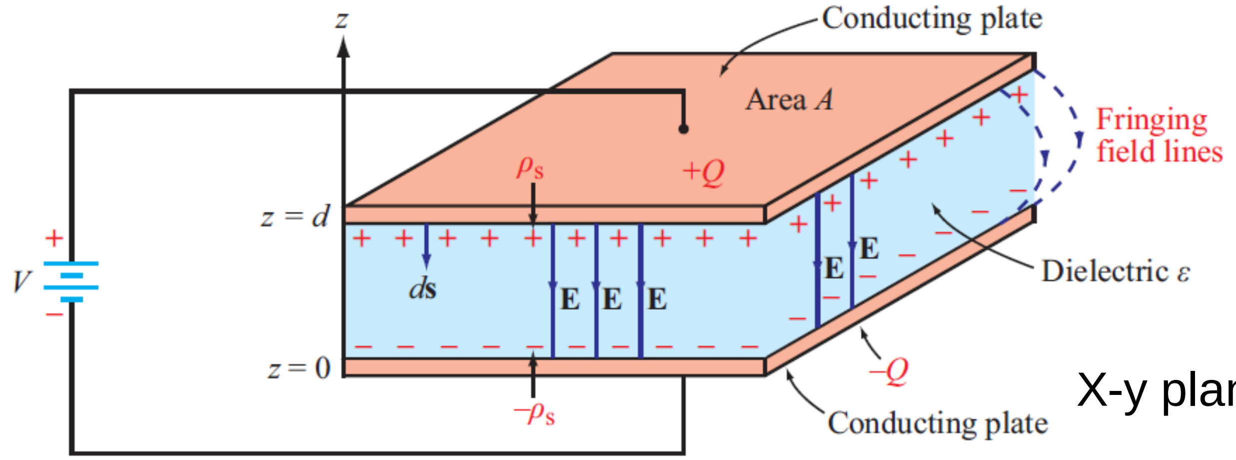
For any resistor:

$$R = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} \qquad (\Omega).$$

$$RC = \frac{\varepsilon}{\sigma}.$$



Example (M)



X-y plane

$$V = -\int_0^d \mathbf{E} \cdot d\mathbf{l} = -\int_0^d (-\hat{\mathbf{z}}E) \cdot \hat{\mathbf{z}}\, dz = Ed,$$

and the capacitance is

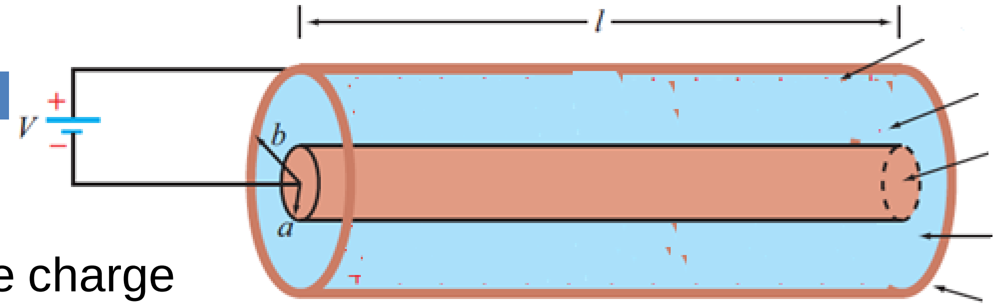
$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d},$$

$\rho s = \varepsilon E$
also
$\rho s = Q/A$
and
$V = E\, d$

Due to boundary condition between Dielectric and conductor

Refer to Notes

Example (N)

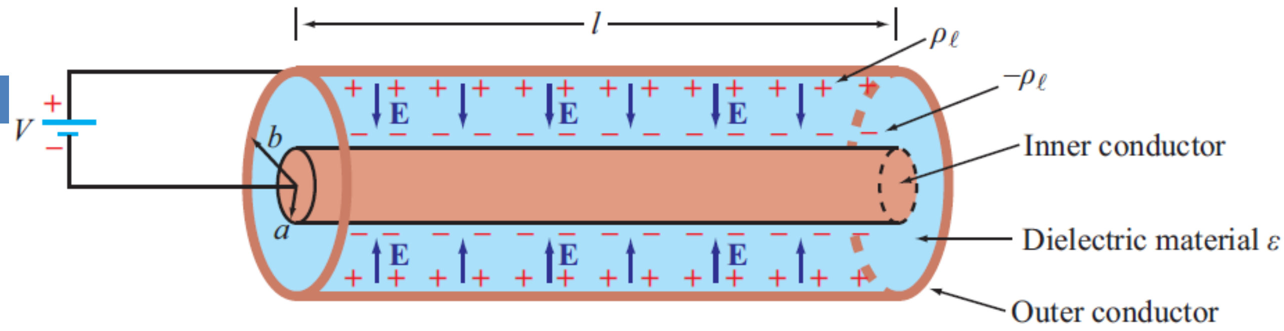


Show all the charges, line charge
Densities, and E fields.

Is E inward to the conductor or outward?

Q is total charge on inside of outer cylinder, and –Q is on outside surface of inner cylinder

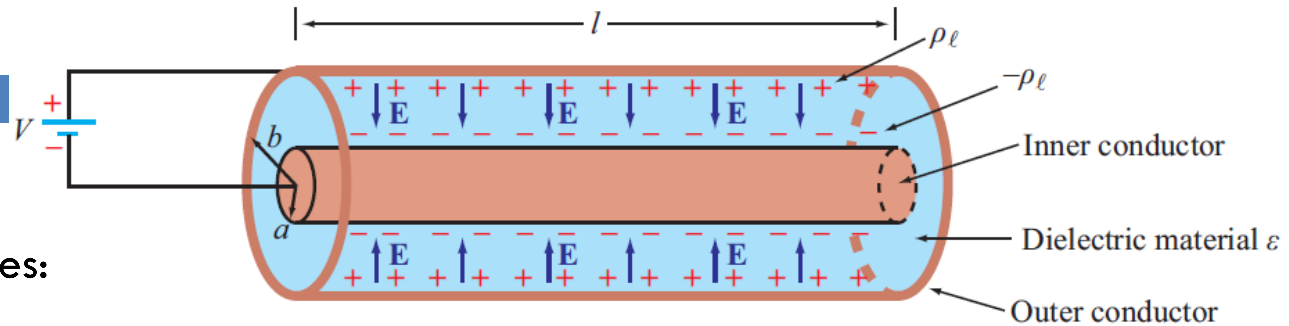
Example (N)



E field is identical at all points on the surface
Directed radially inward!

Q is total charge on inside of outer cylinder, and –Q is on outside surface of inner cylinder

Example (N)
Coaxial Capacitor



Application of Gauss's law gives:

$$\mathbf{E} = -\hat{\mathbf{r}} \frac{Q}{2\pi \varepsilon r l} .$$

The potential difference V between the outer and inner conductors is

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = -\int_a^b \left(-\hat{\mathbf{r}} \frac{Q}{2\pi \varepsilon r l} \right) \cdot (\hat{\mathbf{r}}\, dr)$$

$$= \frac{Q}{2\pi \varepsilon l} \ln \left(\frac{b}{a} \right) .$$

The capacitance C is then given by

$$C = \frac{Q}{V} = \frac{2\pi \varepsilon l}{\ln(b/a)} ,$$

and the capacitance per unit length of the coaxial line is

$$C' = \frac{C}{l} = \frac{2\pi \varepsilon}{\ln(b/a)} \qquad \text{(F/m)}.$$

Remember:

Total Charge = $\int_{z=0}^{h} \int_{\phi=0}^{2\pi} \hat{\mathbf{r}} D_r \cdot \hat{\mathbf{r}} r \, d\phi \, dz =$ Q

or

$2\pi h D_r r =$ Q

What is the break down voltage?

Vbr = Eds x d

Electrostatic Potential Energy

- Assume R(dielectric) =0, σ_conductor = INF, σ_dielectric = very low
- →No current passes through the dielectric and no ohmic loss
- → What happens to the energy?
- STORED! How much? **We (J)** →In form of electrostatic potential energy
  - This is the electric field between the two plates within the dielectric
- We also have the following:
  - C = εA/d & V = Ed & Vol=Area x h
  - We = ½ εE^2 (Vol) (J)

$$W_e = \frac{1}{2} C V^2 \qquad \text{(J)}.$$

$$w_e = \frac{W_e}{v} = \frac{1}{2}\,\varepsilon E^2 \qquad (\text{J/m}^3).$$

Electrostatic potential energy density (Joules/volume)

Example

- Calculate the amount of work performed to transfer total charge Q between the plates in a 1F capacitor .

- $C = Q/V \rightarrow V = Q/C$
- $We = \frac{1}{2} CV^2 \rightarrow \frac{1}{2} Q^2 / C$ (J)

Electrostatic Potential Energy

Energy stored in a capacitor

$$W_e = \tfrac{1}{2} C V^2 \qquad \text{(J)}.$$

$$w_e = \frac{W_e}{\mathcal{V}} = \frac{1}{2}\,\varepsilon E^2 \qquad (\text{J/m}^3).$$

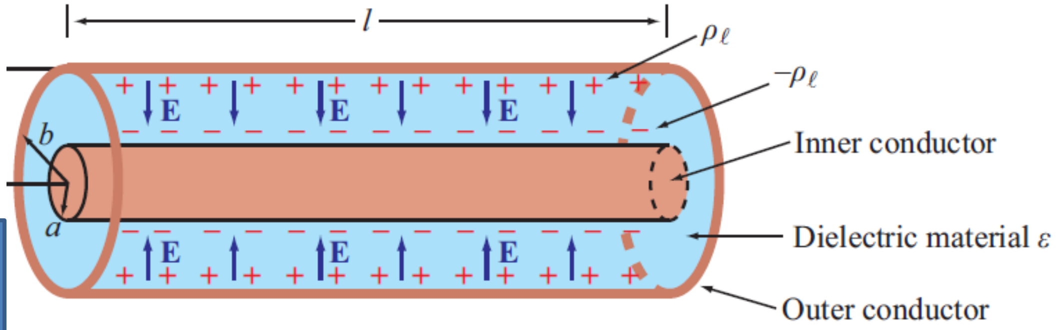
NOTE: Total electrostatic energy stored in any volume V

$$W_e = \frac{1}{2} \int_{\mathcal{V}} \varepsilon E^2 \, d\mathcal{V} \qquad \text{(J)}$$

Example:
Calculate the total stored Energy

Example

The radii of the inner and outer conductors of a coaxial cable are 2 cm and 5 cm, respectively, and the insulating material between them has a relative permittivity of 4. The charge density on the outer conductor is $\rho_l = 10^{-4}$ (C/m)

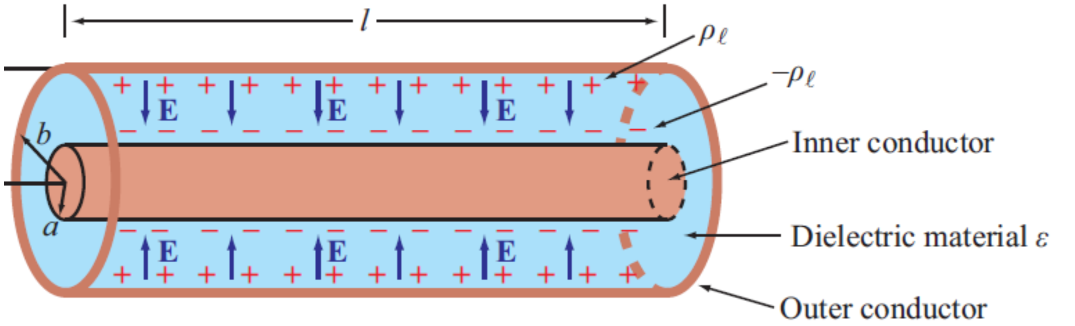


Find E

Find the total energy stored in the volume (dielectric)

Example:
Calculate the total stored Energy

Example The radii of the inner and outer conductors of a coaxial cable are 2 cm and 5 cm, respectively, and the insulating material between them has a relative permittivity of 4. The charge density on the outer conductor is $\rho_l = 10^{-4}$ (C/m)



$$E = \frac{\rho_l}{2\pi\varepsilon r}$$

E around the conductor

$$W_e = \frac{1}{2}\int_{\mathcal{V}} \varepsilon E^2 \, d\mathcal{V}$$

$$= \frac{1}{2}\varepsilon l \int_{r=2\text{ cm}}^{5\text{ cm}} E^2 (2\pi r \, dr)$$

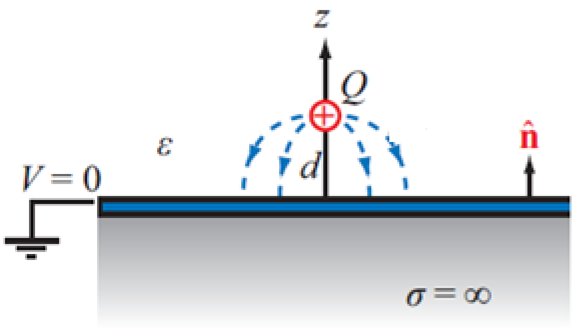
$$= \pi\varepsilon l \int_{2\text{ cm}}^{5\text{ cm}} \left(\frac{\rho_l}{2\pi\varepsilon r}\right)^2 r \, dr$$

$$= \frac{\rho_l^2 l}{4\pi\varepsilon} \int_{2\text{ cm}}^{5\text{ cm}} \frac{dr}{r}$$

$$= \frac{\rho_l^2 l}{4\pi\varepsilon} \ln\left(\frac{5}{2}\right) = 4.1 \quad \text{(J)}.$$

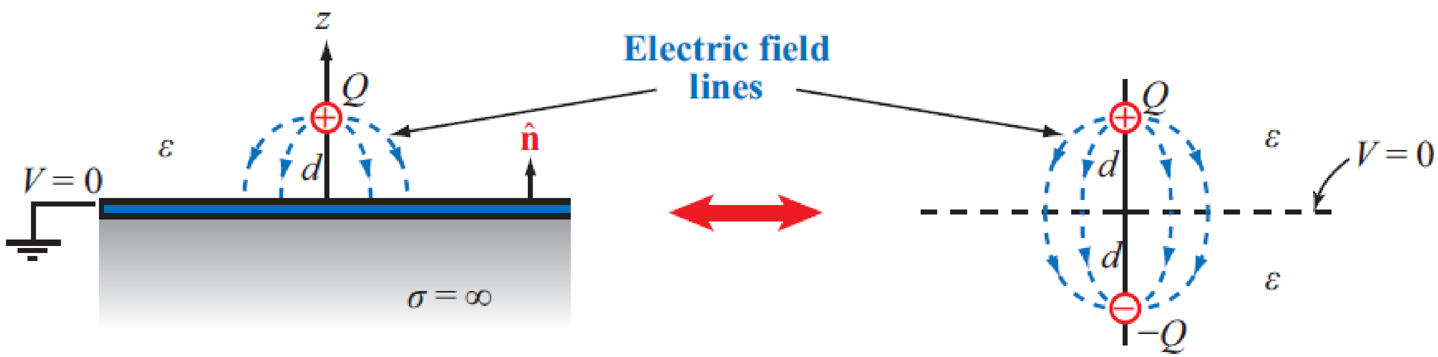
Finding E

- How can we find the total ?
- Assume boundary conditions
  - Et1=Et2=0
  - En1=ρs/ε
- Coulomb's law:
  - Non-uniform distribution of charges
- Gauss's law:
  - Only En1 exists! How do we find them?
- E = - grad V:
  - Mathematically complex!
- So what do we do?



(a) Charge Q above grounded plane

Image Method



(a) Charge Q above grounded plane

(b) Equivalent configuration

Image method simplifies calculation for **E** and *V* due to charges near conducting planes.

1. For each charge Q, add an image charge –Q
2. Remove conducting plane
3. Calculate field due to all charges

Image Method

- It turns out the same thing is true about charge distribution
  - ρl and ρs

Example

Use image theory to determine $\mathbf{E}$ at an arbitrary point $P = (x, y, z)$ in the region $z > 0$ due to a charge Q in free space at a distance d above a grounded conducting plate residing in the $z = 0$ plane.

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 \\ &= \frac{1}{4\pi\varepsilon}\left[\frac{q_1(\mathbf{R}-\mathbf{R}_1)}{|\mathbf{R}-\mathbf{R}_1|^3} + \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{|\mathbf{R}-\mathbf{R}_2|^3}\right].\end{aligned}$$

Example

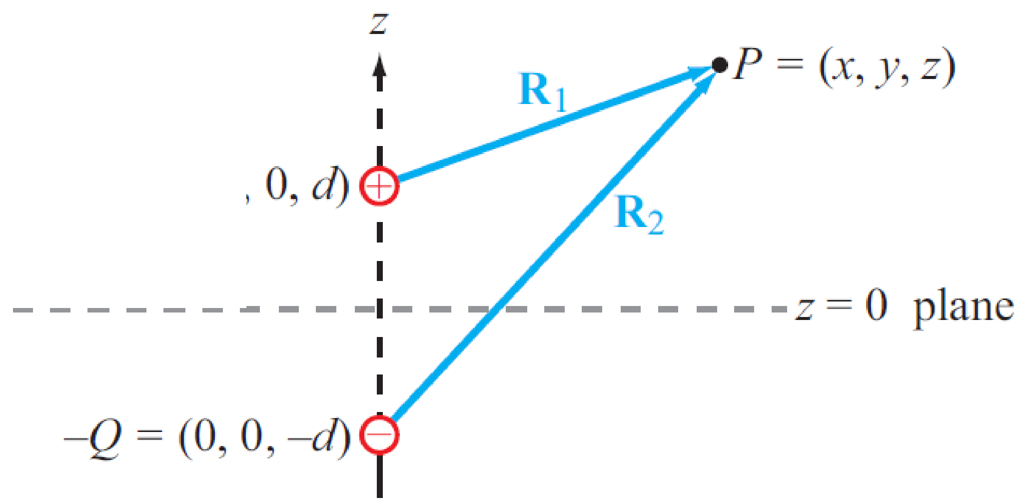
Use image theory to determine $\mathbf{E}$ at an arbitrary point $P = (x, y, z)$ in the region $z > 0$ due to a charge Q in free space at a distance d above a grounded conducting plate residing in the $z = 0$ plane.

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{1}{4\pi\varepsilon}\left[\frac{q_1(\mathbf{R}-\mathbf{R}_1)}{|\mathbf{R}-\mathbf{R}_1|^3} + \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{|\mathbf{R}-\mathbf{R}_2|^3}\right].$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q\mathbf{R}_1}{R_1^3} + \frac{-Q\mathbf{R}_2}{R_2^3}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

for $z \geq 0$.



So how much charge density on the conductor?

Cont.

- Using boundary conditions (assuming free space):

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

$$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s.$$

Z is the normal component
Evaluated at Z=0

$$E_{1n} = -\frac{2Qd}{4\pi\varepsilon_0}\cdot\frac{1}{(x^2+y^2+d^2)^{3/2}}.$$

$$\rho_s = \varepsilon_0 E_{1n} = -\frac{Qd}{2\pi(x^2+y^2+d^2)^{3/2}}.$$