5. MAGNETOSTATICS

Electric vs Magnetic Comparison

Attributes of electrostatics and magnetostatics.

Attribute	Electrostatics
Sources	Stationary charges ρ_v
Fields and Fluxes	$\mathbf{E}$ and $\mathbf{D}$
Constitutive parameter(s)	ε and σ
Governing equations	
• **Differential form**	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$
• **Integral form**	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$
Circuit element(s)	C and R

Most dielectrics μ = μo excluding ferromagnetic materials

J=σE

Gauss's Law

E field is conservative

Gauss's law (integral)

Conservative E field

Electric potential and E

Electrostatic energy density (J/m^3)

Electric Force due to charge (N) – parallel E

CV = Q
R C= ε/σ

Geometry

Electric vs Magnetic Comparison

Attributes of electrostatics and magnetostatics.

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and Fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations • Differential form	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• Integral form	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

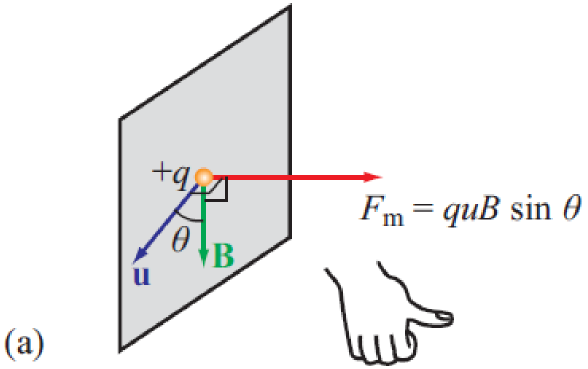
Parallel Properties

Electric & Magnetic Forces

Magnetic force

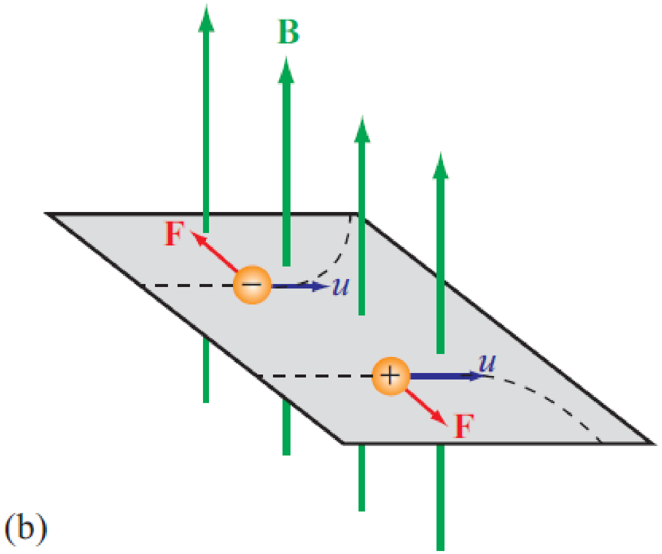
$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad (N)$$

B (Tesla)
U indicates the velocity of the moving charge
Fm is the force acting on the moving charge
Fm is perpendicular with U and B
Note: Fm = quBsin(θ) → Max when angle is 90!



Electromagnetic (Lorentz) force

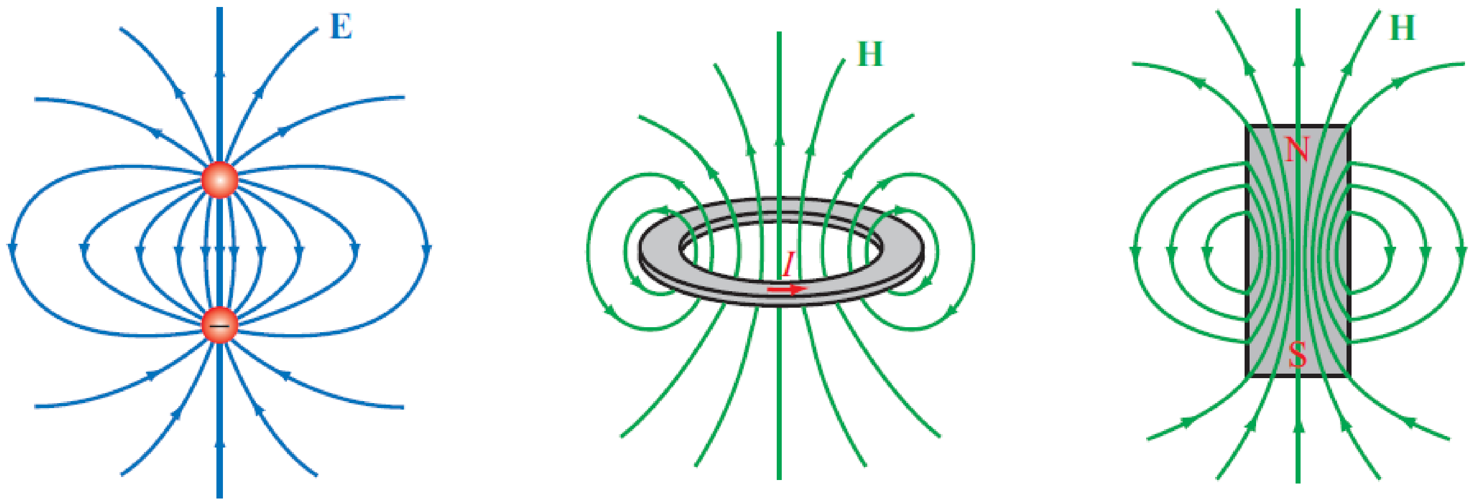
$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$



Important Note:
(q) **dl** = **u** dt (q)→ q/dt (**dl**) = I **dl** = **u** q

The direction of Fm can change depending on the charge!

Magnetic Dipole



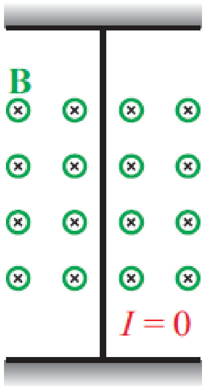
(a) Electric dipole (b) Magnetic dipole (c) Bar magnet

Because a circular loop exhibits a magnetic field pattern similar to the electric field of an electric dipole, it is called a *magnetic dipole*

Magnetic Force on a Current Element

Differential force dFm on a differential current I dl:

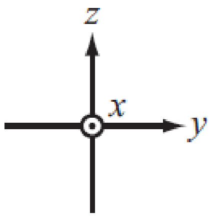
$$d\mathbf{F}_m = I\, d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}.$$



(a)

I Moving +z!
Fm?

I moving -z!
Fm?



Important Note:
(q) **dl** = **u** dt (q)→ q/dt (**dl**) = I **dl** = **u** q

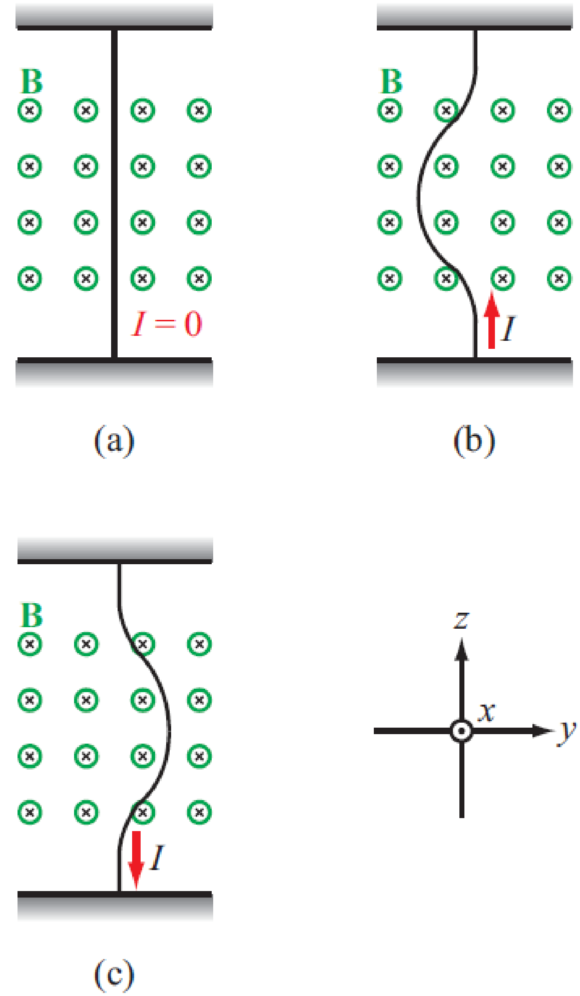
Magnetic Force on a Current Element

Differential force dFm on a differential current I dl:

$$d\mathbf{F}_\mathrm{m} = I\, d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}.$$

For a closed circuit of contour C carrying a current I, the total magnetic force is

$$\mathbf{F}_\mathrm{m} = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}.$$



If a closed wire loop resides in External B

Vector sum
of infinitesimal
vectors dl
Over closed contour C

$$\mathbf{F}_\mathrm{m} = I \left(\oint_C d\mathbf{l} \right) \times \mathbf{B} = 0.$$

Important Note:
(q) **dl** = **u** dt (q)→ q/dt (**dl**) = I **dl** = **u** q

Biot-Savart Law

Magnetic field induced by
a differential current:

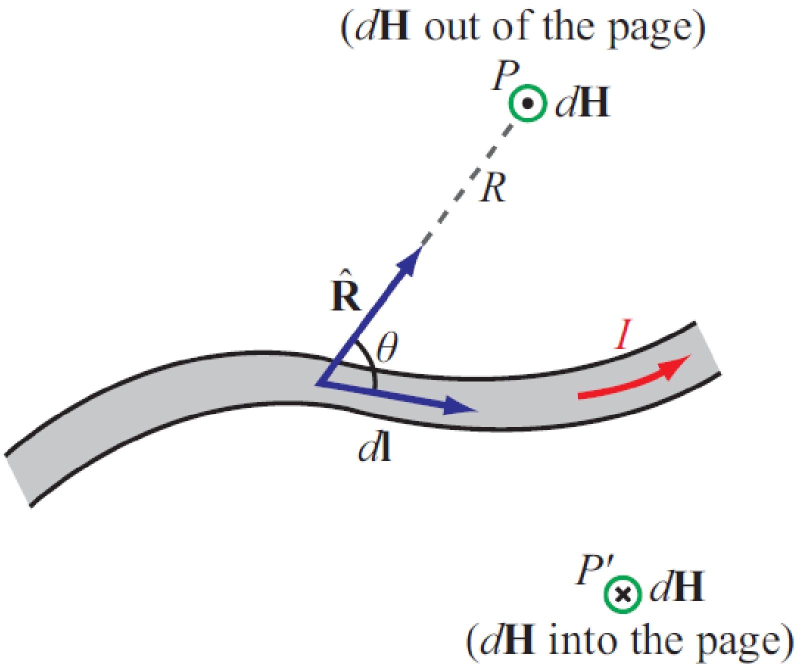
$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$

Hans Oersted demonstrated that steady current induces H
Thus, differential H proportional to steady current flowing through differential vector length
R is the distance vector between dl and the observation point!

For the entire length:

$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)},$$

where l is the line path along which I exists.



R^ = R~/|R|

Magnetic Field due to Current Densities

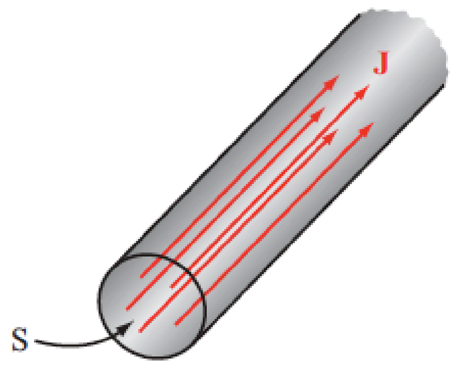
$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)},$$

where l is the line path along which I exists.

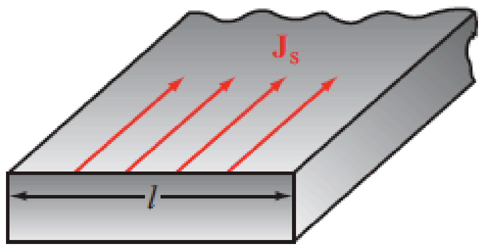
one

$$\mathbf{H} = \frac{1}{4\pi} \int_S \frac{\mathbf{J}_s \times \hat{\mathbf{R}}}{R^2}\, ds \qquad \text{(surface current)},$$

$$\mathbf{H} = \frac{1}{4\pi} \int_V \frac{\mathbf{J} \times \hat{\mathbf{R}}}{R^2}\, dV \qquad \text{(volume current)}.$$



(a) Volume current density J in A/m²



(b) Surface current density $\mathbf{J}_s$ in A/m

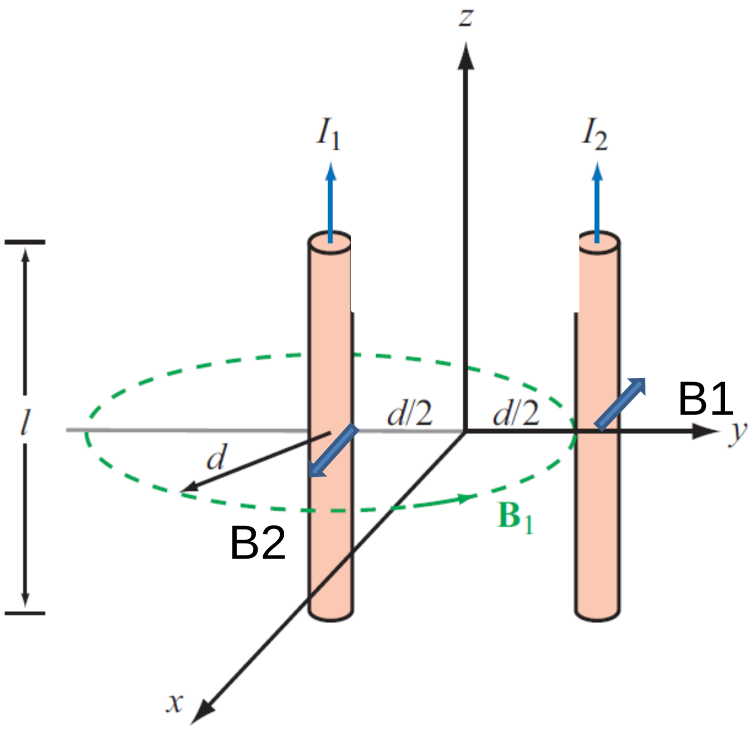
Forces on Parallel Conductors

$$d\mathbf{F}_{\mathrm{m}} = I\ d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}.$$

External B

Which direction is F1?

What about F2?



Forces on Parallel Conductors

$$\mathbf{B}_1 = -\hat{\mathbf{x}}\,\frac{\mu_0 I_1}{2\pi d}\,.$$

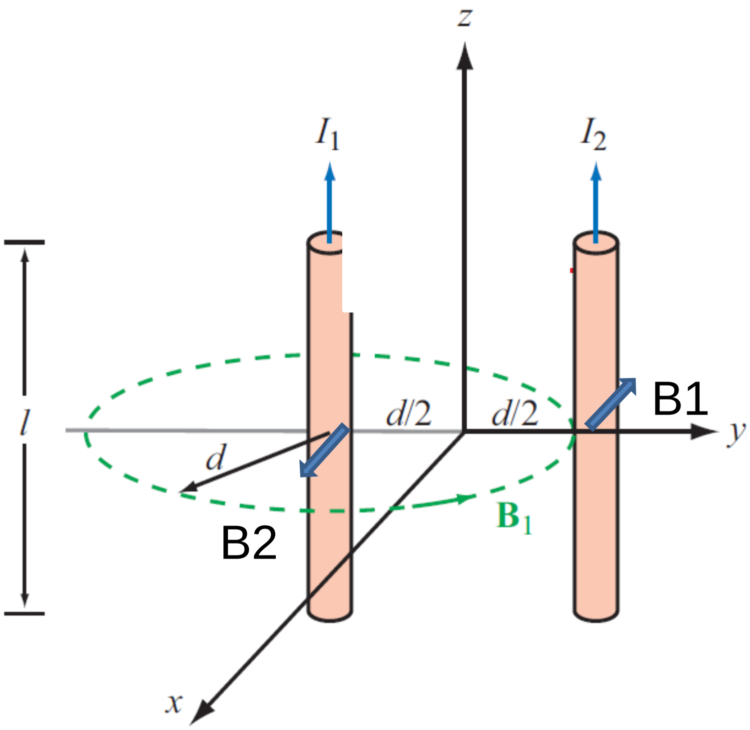
$$\mathbf{F}_2 = I_2 l\hat{\mathbf{z}} \times \mathbf{B}_1 = I_2 l\hat{\mathbf{z}} \times (-\hat{\mathbf{x}})\,\frac{\mu_0 I_1}{2\pi d}$$
$$= -\hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2 l}{2\pi d}\,,$$

and the corresponding force per unit length is

$$\mathbf{F}'_2 = \frac{\mathbf{F}_2}{l} = -\hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}\,.$$

A similar analysis performed for the force per unit length exerted on the wire carrying I_1 leads to

$$\mathbf{F}'_1 = \hat{\mathbf{y}}\,\frac{\mu_0 I_1 I_2}{2\pi d}\,.$$

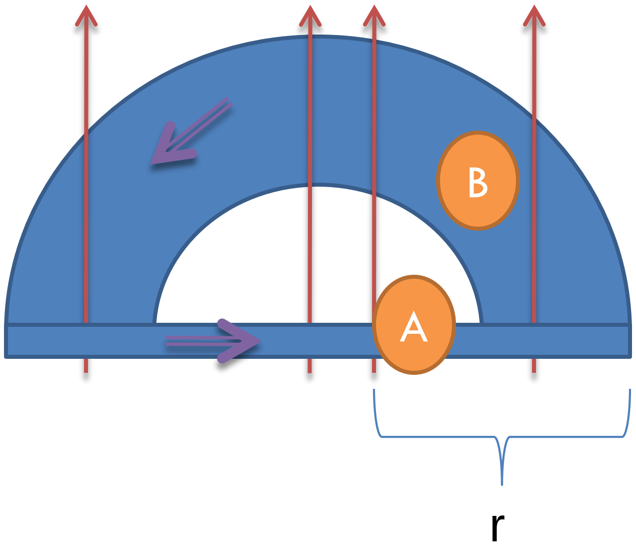


Parallel wires attract if their currents ar
direction, and repel if currents are in o

Notes
A1

Example

- What is B vector?
- Find F at pt A and B



Notes
A2

Find the Differential Magnetic Field Along an Infinitely Long Conductor Carrying I

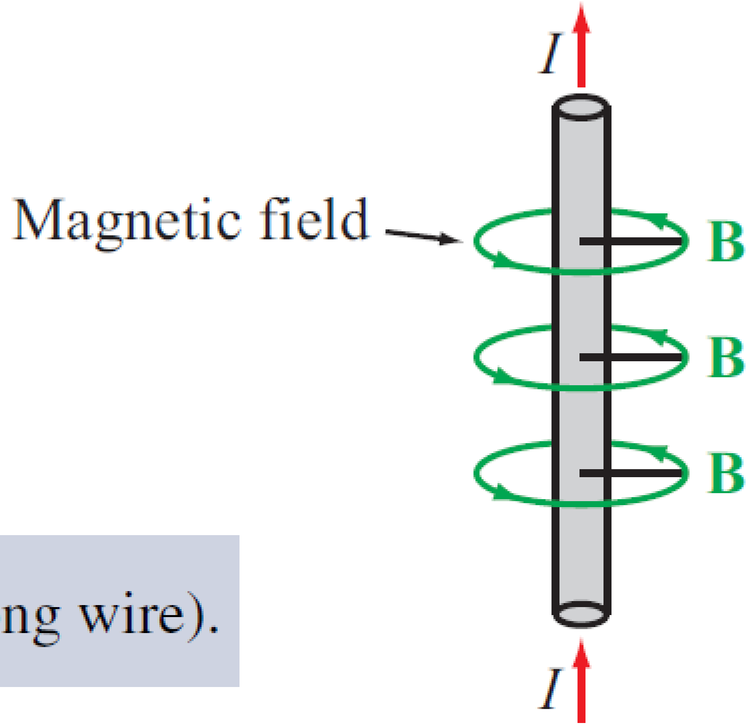
$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$

we find dH in x,y,z
What is the direction of dH?

Notes
A3

Magnetic Field of Long Conductor

$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$



$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \qquad \text{(infinitely long wire).}$$

Maxwell's Equstions

Maxwell's Equations for Electrostatics

Name	Differential Form	Integral Form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$
Kirchhoff's law	$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$

Magnostatic Filed Properties

- Gauss's Law
  - Electrostatic
- Ampere's Law
  - Electrostatic

Governing equations

- Differential form

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \times \mathbf{E} = 0$$

- Integral form

$$\oint_S \mathbf{D} \cdot ds = Q$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

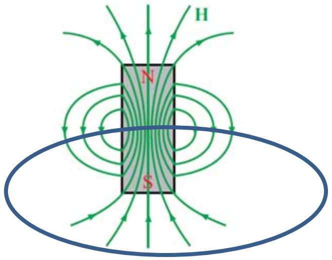
$$\nabla \cdot \mathbf{B} = 0$$

There is no single magnetic dipole

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\oint_S \mathbf{B} \cdot ds = 0$$

Gauss's Law for Magnetic



Net magnetic flux is zero

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I$$

Total Current Passing Through Surface S (open surface)

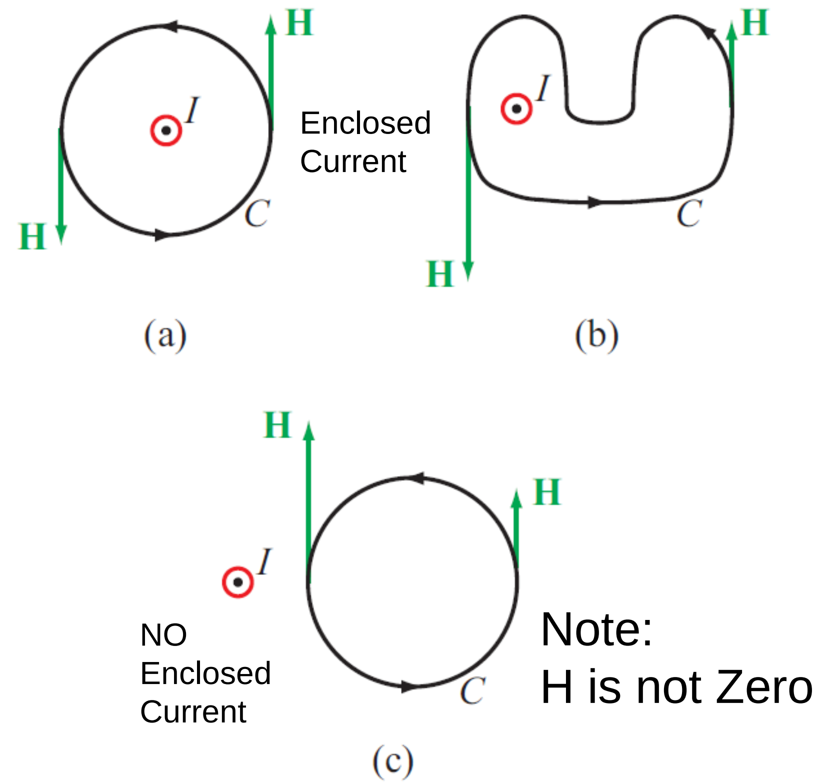
Ampère's Law

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \longleftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}.$$

Open Surface

Convection Current Density



the line integral of **H** is zero for the contour in (c) because the current I (denoted by the symbol ⊙) is not enclosed by the contour C.

Magnetic Properties of Materials

- Depends on how the magnetic dipole moments (**m**) of atoms are changed due to external magnetic field
  - **m** = I A (current * area of the loop) with direction based on right-hand-rule
- Magnetization in materials is due to
  - Orbital motions → orbital magnetic moment (m_o)
    - Most materials are nonmagnetic when there is no magnetic field → atom are randomly oriented → very small net magnetic moment
  - Electron spin → spin magnetic moment (m_s)
    - If odd number of e → unpaired electrons → net nonzero m_s
- Three type of materials
  - Diamagnetic → (de / out of)
  - Paramagnetic→ (para is near)
  - Ferromagnetic → (ferrum or iron)

Magnetic Permeability

M = Magnetization Vector = vector sum of magnetic dipole moments

$$\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{M} = \mu_0(\mathbf{H} + \mathbf{M})$$

Magnetic susceptibility

$$\mathbf{M} = \chi_\mathrm{m}\mathbf{H}$$

$$\mathbf{B} = \mu_0(\mathbf{H} + \chi_\mathrm{m}\mathbf{H}) = \mu_0(1 + \chi_\mathrm{m})\mathbf{H},$$

$$\mathbf{B} = \mu\mathbf{H},$$

Magnetic permeability

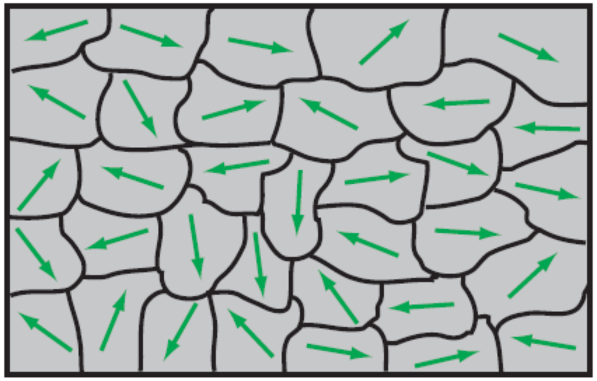
Properties of magnetic materials.

	Diamagnetism	Paramagnetism	Ferromagnetism
Permanent magnetic dipole moment	No	Yes, but weak	Yes, and strong
Primary magnetization mechanism	Electron orbital magnetic moment	Electron spin magnetic moment	Magnetized domains
Direction of induced magnetic field (relative to external field)	Opposite	Same	Hysteresis
Common substances	Bismuth, copper, diamond, gold, lead, mercury, silver, silicon	Aluminum, calcium, chromium, magnesium, niobium, platinum, tungsten	Iron, nickel, cobalt
Typical value of χ_m **Typical value of μ_r**	$\approx -10^{-5}$ ≈ 1	$\approx 10^{-5}$ ≈ 1	$\lvert\chi_m\rvert \gg 1$ and hysteretic $\lvert\mu_r\rvert \gg 1$ and hysteretic

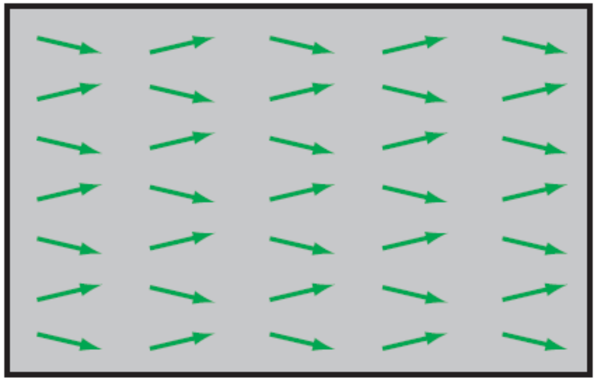
$$\mathbf{B} = \mu_0(\mathbf{H} + \chi_m \mathbf{H}) = \mu_0(1 + \chi_m)\mathbf{H},$$

$$\mathbf{B} = \mu \mathbf{H},$$

Magnetic Hysteresis



(a) Unmagnetized domains



(b) Magnetized domains

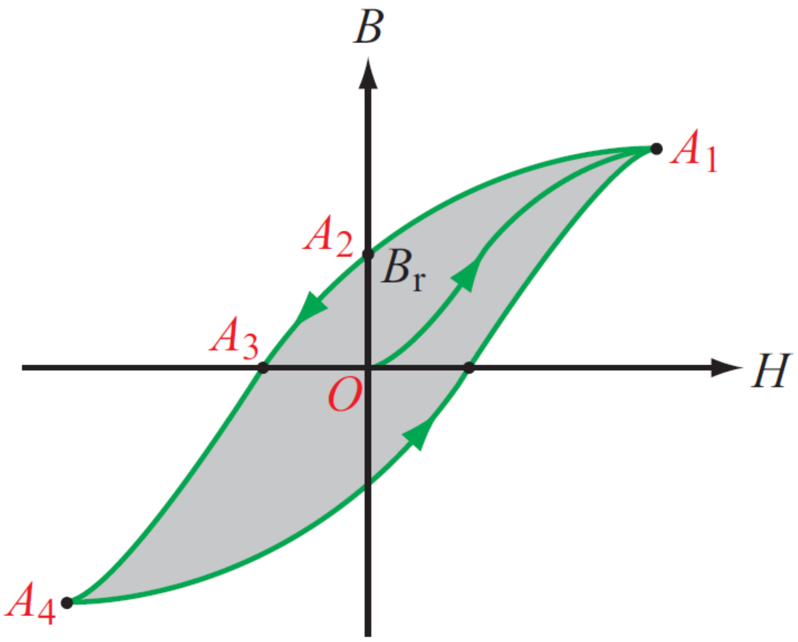
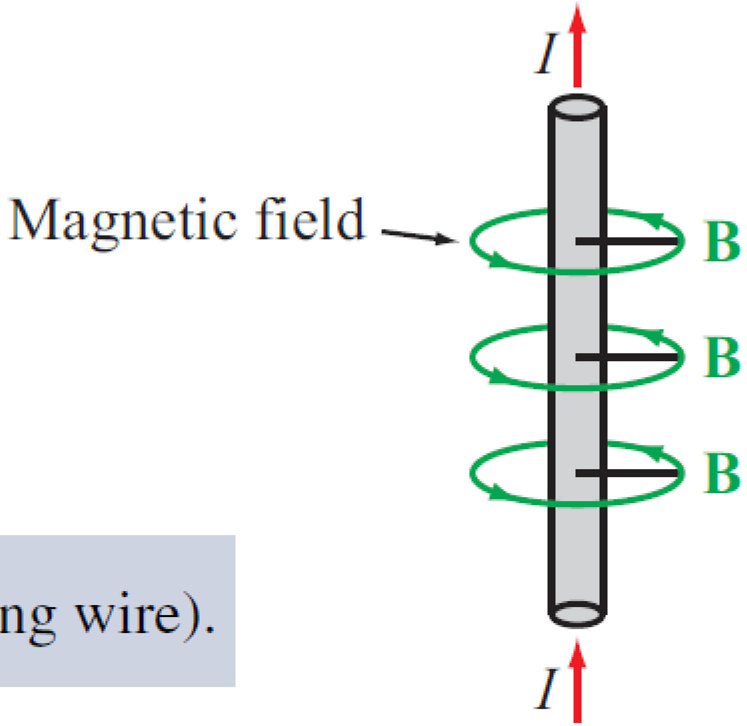


Figure 5-22: Typical hysteresis curve for a ferromagnetic material.

Magnetic Field of Long Conductor

$$d\mathbf{H} = \frac{I}{4\pi}\frac{d\mathbf{l}\times\hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$



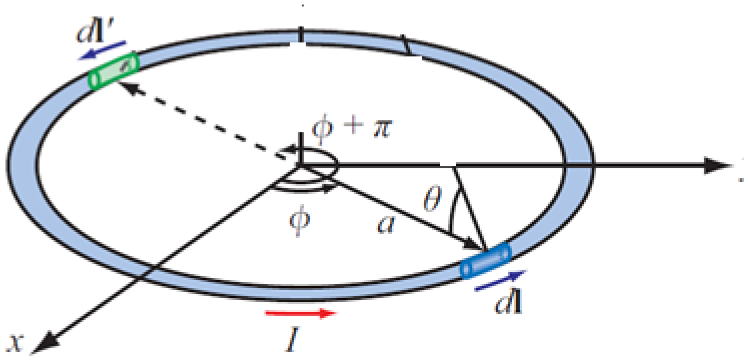
$$\mathbf{B} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi r} \qquad \text{(infinitely long wire).}$$

Notes
B0 – Prove!

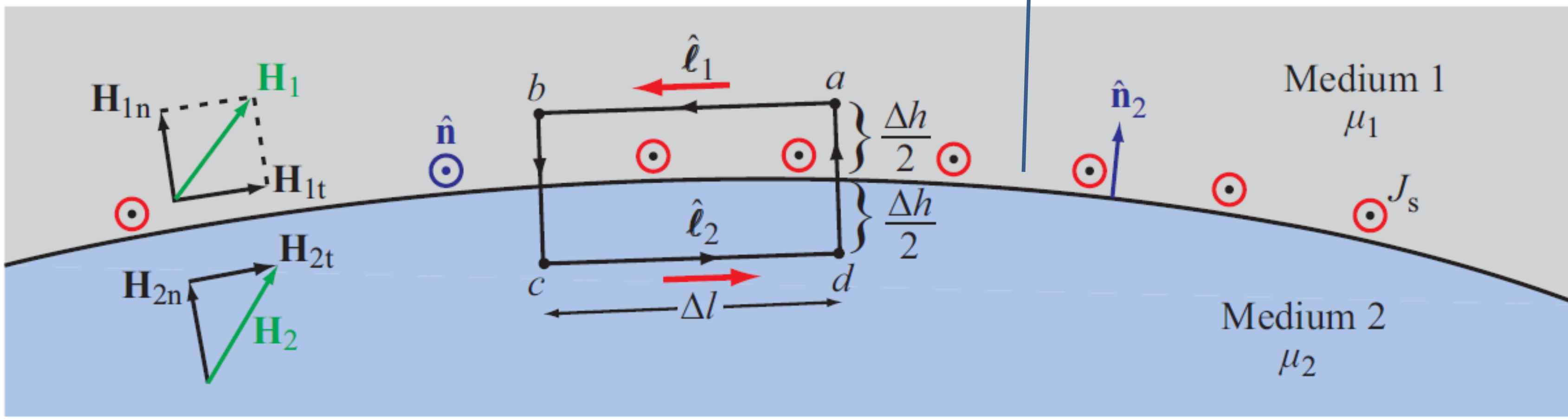
Assuming a loop carrying current – Find the magnetic field

- Do it on your own!
  - Compare it with the previous case!
  - What happens if you are in the middle of the loop?

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I a^2}{2(a^2 + z^2)^{3/2}} \qquad \text{(A/m)}$$



Boundary Conditions



Electrostatic

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \rightarrow D_{1n} - D_{2n} = \rho_s.$$

$$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s.$$

By analogy, application of Gauss's law for magnetism, as expressed by Eq. (5.44), leads to the conclusion that

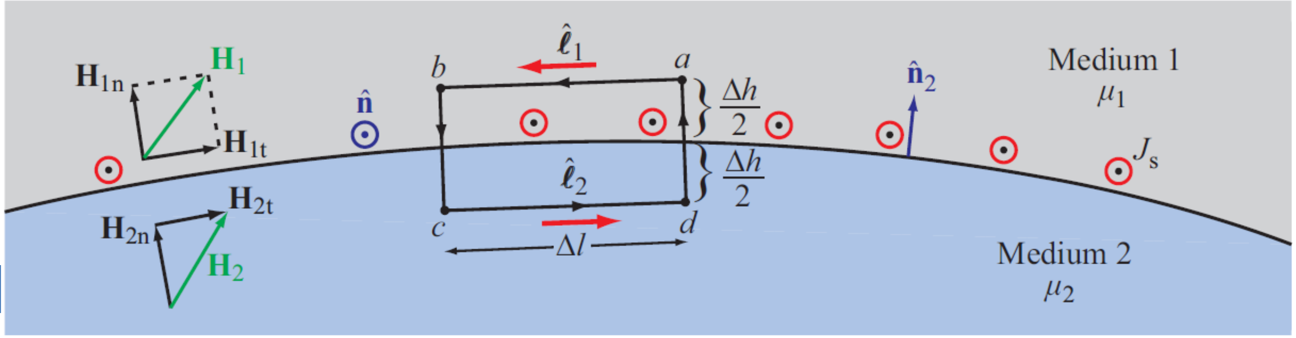
$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \rightarrow B_{1n} = B_{2n}.$$

Thus the normal component of **B** *is continuous across the boundary between two adjacent media.*

Surface currents can exist only on the surfaces of perfect conductors and superconductors. Hence, ***at the interface between media with finite conductivities,*** $\mathbf{J}_s = 0$ and

$$H_{1t} = H_{2t}.$$

Example:



- Determine the angle between H1 and ^n2 = ^z if H2 = (^x3+^z2) (A/m), mr1 = 2, and mr2 = 8, and Js = 0.
  - Find H1t
  - Find H1n

$$\mathbf{H}_2 = \hat{\mathbf{x}}3 + \hat{\mathbf{z}}2$$

$$H_{1x} = H_{2x} = 3$$

$$\mu_1 H_{1z} = \mu_2 H_{2z}$$

$$H_{1z} = \frac{\mu_2}{\mu_1} H_{2z} = \frac{8}{2} \times 2 = 8$$

$$\mathbf{H}_1 = \hat{\mathbf{x}}3 + \hat{\mathbf{z}}8$$

$$\mathbf{H}_1 \cdot \hat{\mathbf{z}} = H_1 \cos\theta$$

$$\cos\theta = \frac{\mathbf{H}_1 \cdot \hat{\mathbf{z}}}{H_1} = \frac{8}{\sqrt{9+64}} = \frac{8}{\sqrt{73}} = 0.936$$

$$\theta = 20.6^\circ.$$

Notes B1

Magnetic Flux

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \longleftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

Note:
Tesla = N / A.m = Wb /m^2
uo has unit of H/m
H = Wb /A

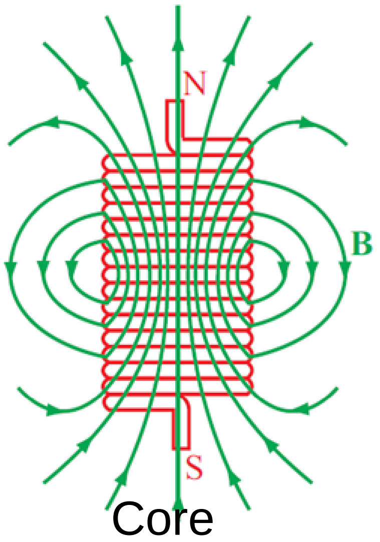
Magnetic Flux

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(Wb).}$$

Can be positive or negative

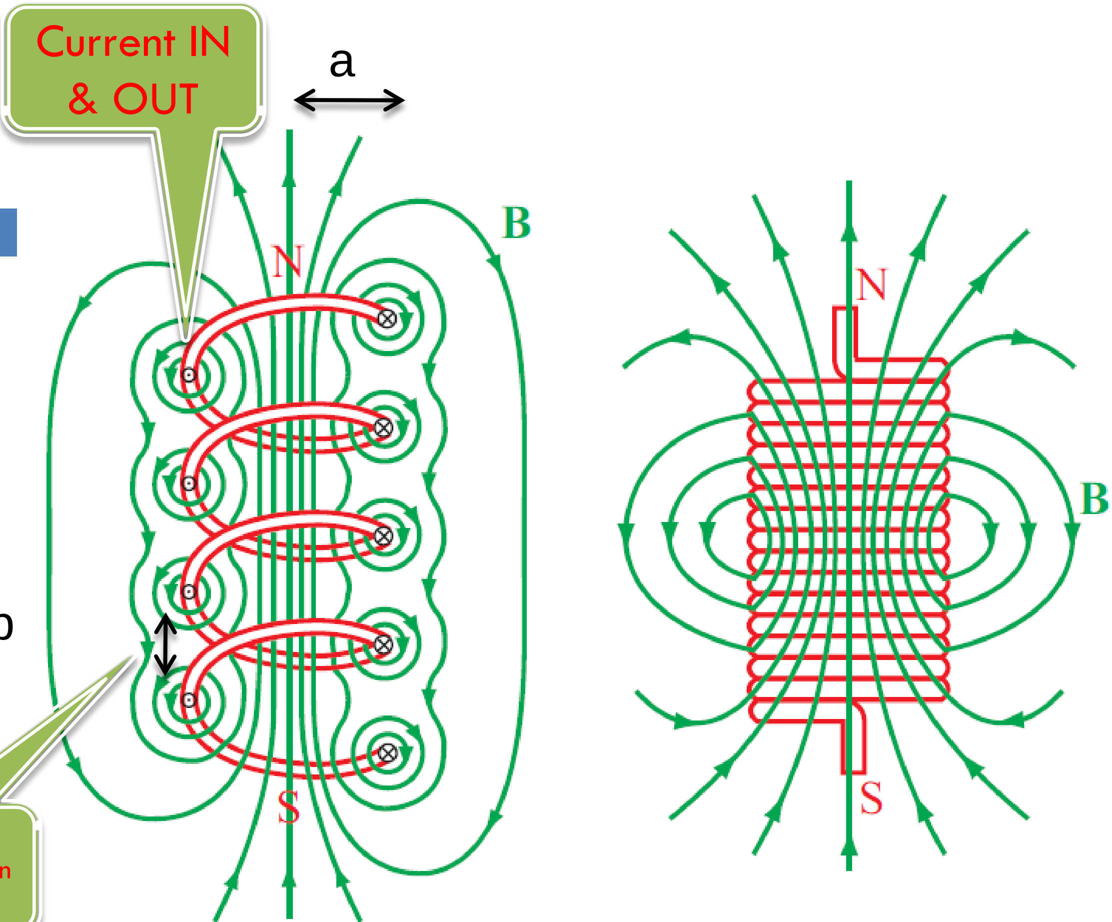
Notes
B2

Inductors



- Capacitors
  - Store Energy in E Field between plates
- Inductors
  - Generate uniform magnetic field when I is passing
  - Store Energy in H field along the current carrying conductor
- Inductance is related to magnetic flux and current
  - Self-inductance
  - Mutual inductance (e.g., B12, L12)

Solenoid



(a) Loosely wound solenoid

(b) Tightly wound solenoid

dZ = diff. distance between loops

Inside the solenoid:

n = turns / meter = N/l
N = number of turns
I = current along a single loop
2a = loop diameter

For a single loop

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I a^2}{2(a^2 + z^2)^{3/2}} \quad \text{(A/m)}$$

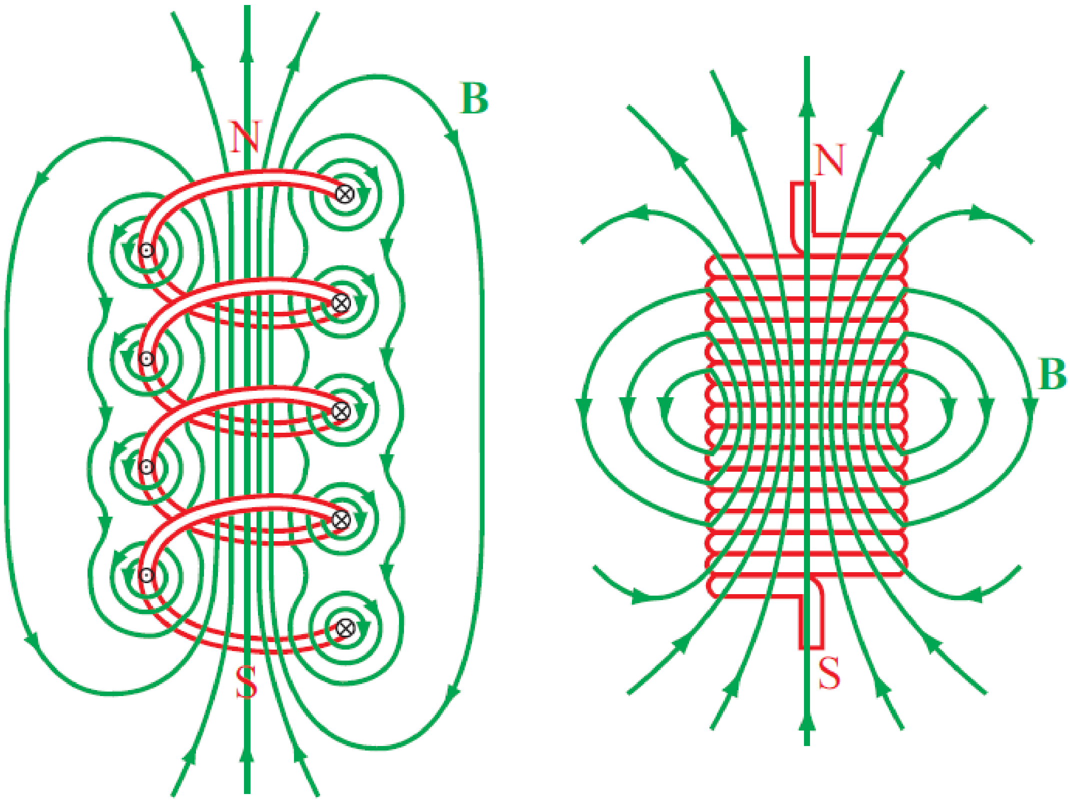
For all N loops

$$\mathbf{H} = \hat{\mathbf{z}} \frac{N I a^2}{2(a^2 + z^2)^{3/2}} \quad \text{(A/m)}$$

→ dB = udH ; N = n . dz

Notes B5

Solenoid



(a) Loosely wound solenoid

(b) Tightly wound solenoid

Inside the solenoid:

$$\mathbf{B} \simeq \hat{\mathbf{z}}\mu n I = \frac{\hat{\mathbf{z}}\mu N I}{l} \quad \text{(long solenoid with } l/a \gg 1\text{)}$$

Notes B5

Inductance

Magnetic Flux

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(Wb).}$$

Flux Linkage

$$\Lambda = N\Phi = \mu \frac{N^2}{l} I S \qquad \text{(Wb)}$$

Inductance

$$L = \frac{\Lambda}{I} \qquad \text{(H).}$$

Solenoid

$$L = \mu \frac{N^2}{l} S \qquad \text{(solenoid),} \qquad (5.95)$$

and for two-conductor configurations similar to those of Fig. 5-27,

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s}. \qquad (5.96)$$