S-DOMAIN ANALYSIS: POLES, ZEROS, AND BODE PLOTS

The main objectiveis to find amplifier voltage gain as a transfer function of the complex frequency *s*. In this s-domain analysis

- a capacitance C is replaced by an admittance *sC,* or equivalently an impedance *1/sC,* and
- an inductance *L* is replaced by an impedance *sL*. Then, using usual circuit-analysis techniques, one derives the voltage transfer function *T(s) = Vo(s)/V,(s).*

Once the transfer function *T(s)* is obtained, it can be evaluated for **physical frequencies** by replacing *s by jω.* The resulting transfer function *T(jω)* is in general a complex quantity whose magnitude gives the magnitude response (or transmission) and whose angle gives the phase response of the amplifier.

In general, for all the circuits dealt with in this chapter, *T(s)* can be expressed in the form

$$T(s)=\frac{a_m s^m + a_{m-1}s^{m-1}+\ldots+a_0}{s^n+b_{n-1}s^{n-1}+\ldots+b_0}$$

where the coefficients *a* and *b* are real numbers and $m \le n$

n is called the **order of the network.**

Any function: $a_m x^m + a_{m-1}x^{m-1}+\ldots+a_0$

Can be presented in the following form:

$$a_m\cdot(x-x_1)\cdot(x-x_2)\cdot\ldots\cdot(x-x_m)$$

where $x_1, x_2 \ldots x_m$ are the roots of $a_m x^m + a_{m-1}x^{m-1}+\ldots+a_0=0$

For example the Equation: $x^2+4x-5=0$ has the roots

$$x_{1,2}=-\frac{a_1}{2}\pm\sqrt{\left(-\frac{a_1}{2}\right)^2+a_0}=-\frac{4}{2}\pm\sqrt{\left(-\frac{4}{2}\right)^2+5}=-2\pm\sqrt{4+5}=-2\pm3$$

Or $x_1=1$, $x_2=-5$ and therefore

$$(x-1)\cdot(x+5)=x^2-x+5x+(1)\cdot(-5)=x^2+4x-5=0$$

POLES AND ZEROS

An alternate form for expressing *T(s)* is

$$T(s) = a_m \frac{(s - Z_1)\cdot(s - Z_2)\cdot\ldots\cdot(s - Z_m)}{(s - P_1)\cdot(s - P_2)\cdot\ldots\cdot(s - P_n)}$$

where *Z1, Z2,..., Zm*, are the roots of the numerator polynomial, and *P1, P2,..., Pn* are the roots of the denominator polynomial.

- *Z1, Z2,..., Zm*, are called the **transfer-function zeros** or **transmission zeros,**
- *P1, P2,..., Pn* are the **transfer-function poles** or the **natural modes** of the network.

A ***transfer function*** is specified in terms of its poles and zeros

The ***poles*** and ***zeros*** can be either real or complex numbers. However, since the *a* and *b* coefficients are real numbers, the complex poles (or zeros) must occur in **conjugate pairs.** That is, if *5+j3* is a ***Zero***, then *5-j3* also must be a ***Zero***.

For example: $x^2 + 4x + 5 = 0$ has the following roots:

$$x_{1,2} = -\frac{4}{2} \pm \sqrt{\left(-\frac{4}{2}\right)^2 - 5} = -2 \pm \sqrt{4-5} = -2 \pm j$$

- If ***Zero*** is pure imaginary ($\pm j\omega_z$) the transfer function $T(j\omega)$ to be exactly zero at $\omega = \omega_z$. This is because the numerator will have the factors $(s + j\omega_z)(s - j\omega_z) = (s^2 + \omega_z^2)$, which for physical frequencies becomes $(-\omega^2 + \omega_z^2)$, and thus the transfer fraction will be exactly zero at $\omega = \omega_z$.
- If *s* much greater than all the poles and zeros, the transfer function becomes $T(s) \approx a_m/s^{n-m}$. Thus the transfer function has $(n — m)$ zeros at $s = \propto$.

First-Order Functions

The ***first-order*** **transfer functions** can be written as of the general form

$$T(s) = \frac{a_1 s + a_0}{s + \omega_0}$$

where $-\omega_0$ is **pole frequency -** the location of the real pole.
The constants a_0 and a_1 determine the type of STC network.
There are two types of STC networks, low pass and high pass.

For the ***low-pass*** first-order network we have

$$T(s) = \frac{a_0}{s + \omega_0}$$

In this case the **dc gain** is a_0/ω_0, and ω_0 is the ***corner or 3-dB frequency***.
this transfer function has one Zero at $s = \infty$.

The first-order ***high-pass*** transfer function has a zero at dc and can be written as

$$T(s) = \frac{a_1 s}{s + \omega_0}$$

Of specific interest are the plots of the magnitude and phase responses of the two special kinds of STC networks. Such plots can be employed to generate the magnitude and phase plots of a high-order transfer function, as explained below.

A simple technique exists for obtaining an approximate plot of the magnitude and phase of a transfer function given its poles and zeros. The technique is particularly useful in the case of real poles and zeros. The method was developed by H. Bode, and the resulting diagrams are called **Bode plots.**

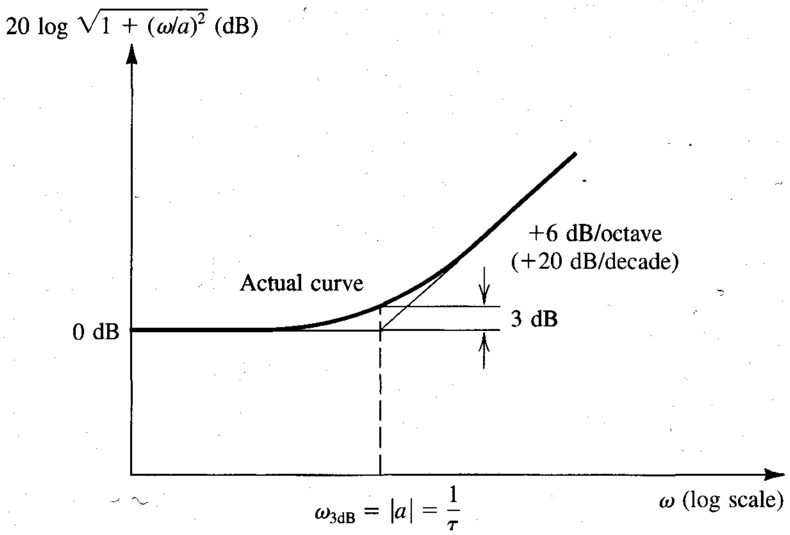
Bode Plots

A transfer function consists of a product of factors of the form $s + a$, where such a factor appears on top if it corresponds to a zero and on the bottom if it corresponds to a pole.

- the magnitude response in decibels of the network can be obtained by summing together terms of the form $20\log\sqrt{a^2+\omega^2}$.
- the phase response can be obtained by summing terms of the form $\tan^{-1}(\omega/a) \equiv \arctan(\omega/a)$.

For convenience we can extract the constant a and write the typical magnitude term in the form $20\log\sqrt{1+(\omega/a)^2}$.

On a plot of decibels versus log frequency this term gives rise to the curve and straight-line asymptotes shown in Figure.



Bode plot for the typical magnitude term. The curve shown applies for the case of a ***Zero****. For a* ***Pole****, the high-frequency asymptote should be drawn with a -6-dB/octave(-20 dB/decade) slope.*

The low-frequency asymptote is a horizontal straight line at 0-dB level and the high-frequency asymptote is a straight line with a slope of 6 dB/octave or, equivalently, 20 dB/decade. The two asymptotes meet at the frequency $\omega = |a|$, which is called the **corner frequency.**

The actual magnitude plot differs slightly from the value given by the asymptotes; the maximum difference is 3 dB and occurs at the comer frequency.

For a = 0 — that is, a pole or a zero at $s = 0$ — the plot is simply a straight line of 6 dB/octave slope intersecting the 0-dB line at $\omega = 1$.

In summary, to obtain the Bode plot for the magnitude of a transfer function, the asymptotic plot for each pole and zero is first drawn. The slope of the high-frequency asymptote of the curve corresponding to a zero is +20 dB/decade, while that for a pole is -20 dB/decade. The various plots are then added together, and the overall curve is shifted vertically by an amount determined by the multiplicative constant of the transfer function.

EXAMPLE

An amplifier has the voltage transfer function

$$T(s) = a_m \frac{10s}{\left(1 + s/10^2\right)\cdot\left(1 + s/10^5\right)}$$

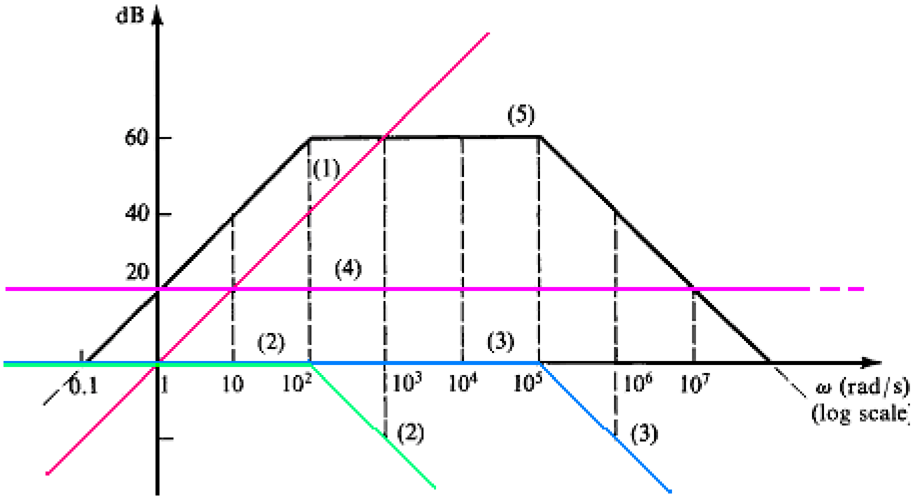
Find the poles and zeros and sketch the magnitude of the gain versus frequency.

Find approximate values for the gain at w = 10, 10^3, and 10^6 rad/s.

Solution

The zeros are as follows: one at $s = 0$ and one at $s = \propto$.

The poles are as follows: one at $s = -10^2$ rad/s and one at $s = -10^5$ rad/s.



Bode plots for Example.

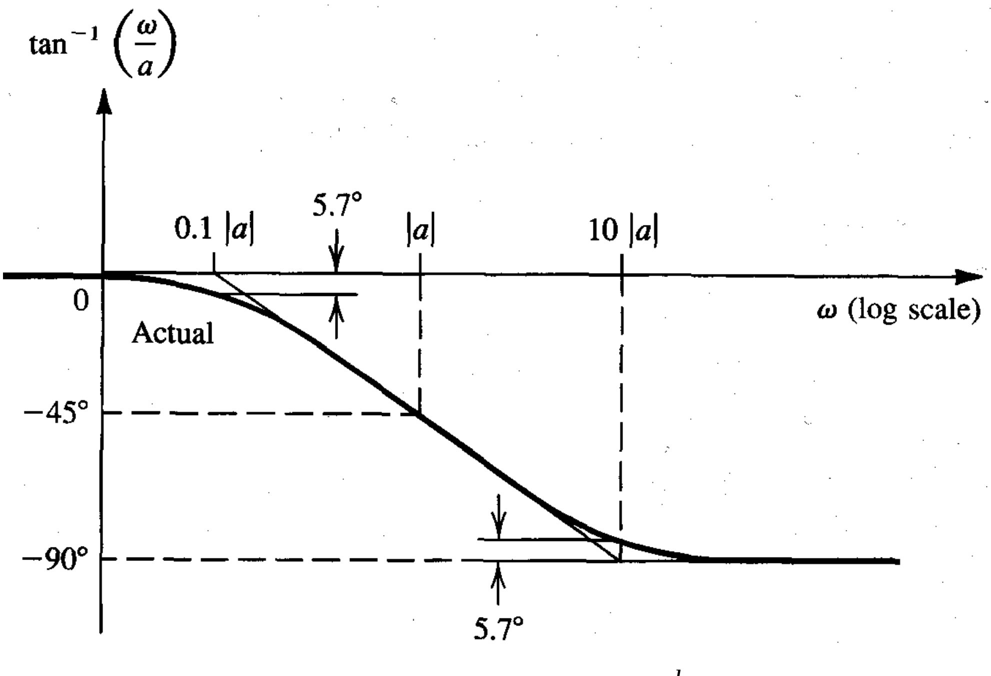
- Curve 1: Straight line with +20 dB/decade slope, corresponds to the *s* term (that is, the zero at s = 0) in the numerator.
- Curve 2: The pole at $s = -10^2$ results in, which consists of two asymptotes intersecting at $\omega = 10^2$.
- Curve 3: The pole at $s = -10^5$ is represented by, where the intersection of the asymptotes is at $\omega = 10^5$.
- Curve 4: The Zero at ($s=\infty$) represents the multiplicative constant of value 10 (+20 dB).

Adding the four curves results in the asymptotic Bode diagram of the amplifier gain (curve 5). Note that since the two poles are widely separated, the gain will be very close to 10^3 (60 dB) over the frequency range 10^2 to 10^5 rad/s. At the two comer frequencies (10^2 and 10^5 rad/s) the gain will be approximately 3 dB below the maximum of 60 dB.

At the three specific frequencies the values of the gain as obtained from the Bode plot and from exact evaluation of the transfer function are as follows:

ω	Approximate	Exact Gain
10	40 dB	39.96 dB
10^3	60 dB	59.96 dB
10^6	40 dB	39.96 dB

Next consider the Bode phase Plot.
The asymptotic plot consists of three straight lines:
Line 1. the horizontal line at $\varphi=0^{\circ}$ and extends up to $\omega=0.1|a|$
Line 2. has a slope -45° per decade and extends from $\omega=0.1|a|$ to $\omega=10|a|$
Line 3. the horizontal line at $\varphi=90^{\circ}$ and extends up to $\omega=10\ |a|$

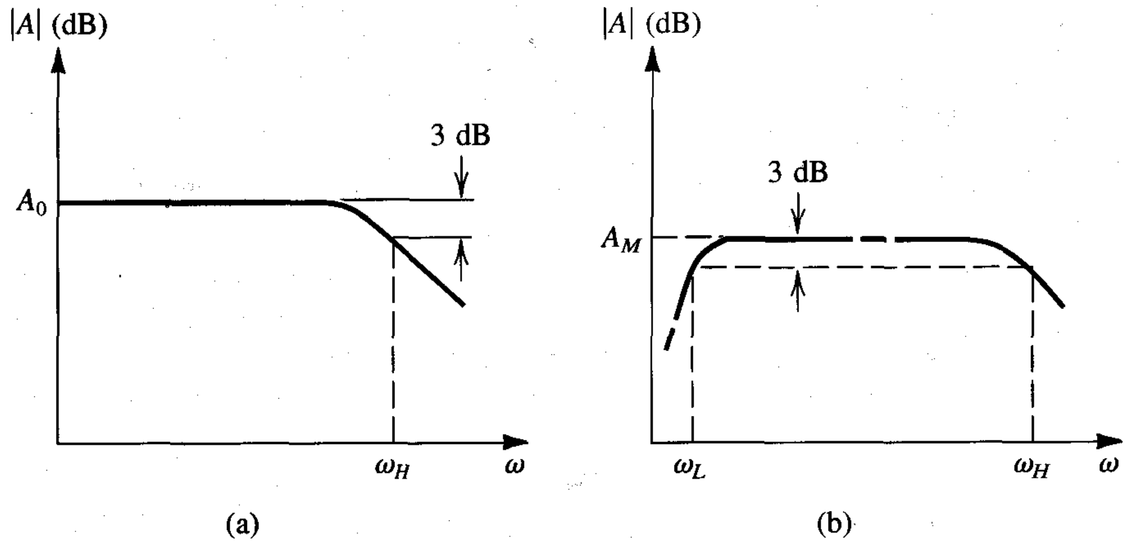


Bode plot of the typical phase term $\tan^{-1}(\omega/a)$ when ***a*** is negative.

THE AMPLIFIER TRANSFER FUNCTION

The Three Frequency Bands

The amplifier gain is almost constant over a wide frequency range called the **midband.** In this frequency range all capacitances (coupling, bypass, and transistor internal capacitances) have negligible effects and can be ignored in gain calculations.



Frequency response for (a) a dc amplifier and (b) a capacitively coupled amplifier. The only difference between the two types is that the gain of the ac amplifier falls off at low frequencies.

At the high-frequency end of the spectrum the gain drops owing to the effect of the internal capacitances of the device. On the other hand, at the low-frequency end of the spectrum the coupling and bypass capacitances no longer act as perfect short circuits and thus cause the gain to drop.

The extent of the midband is usually defined by the two frequencies ω_H and ω_L. These are the frequencies at which the gain drops by 3 dB below the value at midband.

The amplifier **bandwidth** is usually defined as

$$BW = \omega_H - \omega_L$$

The Gain Function *A(s)*

The amplifier gain as a function of the complex frequency д can be expressed in the general form $A(s) = A_M \cdot F_H(s) \cdot F_L(s)$

where $F_L(s)$ and $F_H(s)$ are functions that account for the dependence of gain on frequency in the low-frequency band and in the high-frequency band, respectively and AM is a midband gain.

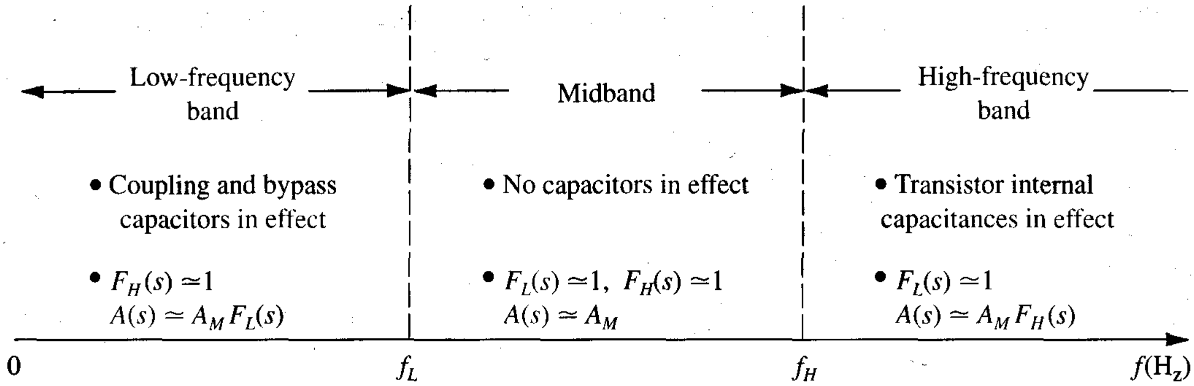
Thus for $\omega_L << \omega << \omega_H$, $A(s) \cong A_M$

the gain of the amplifier in the low-frequency band, $A_L(s)$, can be expressed as

$A_L(s) = A_M \cdot F_L(s)$

and the gain in the high-frequency band can be expressed as

$A_H(s) = A_M \cdot F_H(s)$



The three frequency bands that characterize the frequency response of capacitively coupled amplifiers. For dc amplifiers, the absence of coupling and bypass capacitors causes $F_L(s)$ = 1 and f_L = 0, thus the midband gain extends to zero frequency.

The Low-Frequency Response

The function $F_L(s)$, which characterizes the low-frequency response of the amplifier, takes *the* general form

$$F_L(s) = \frac{(s-\omega_{Z1})\cdot(s-\omega_{Z2})\cdot\ldots\cdot(s-\omega_{ZnL})}{(s-\omega_{P1})\cdot(s-\omega_{P2})\cdot\ldots\cdot(s-\omega_{PnL})}$$ where

$\omega_{Z1}, \omega_{Z2}, \ldots, \omega_{ZnL}$ are positive numbers representing the frequencies of the $\boldsymbol{n_L}$ low-frequency poles

$\omega_{Z1}, \omega_{Z2}, \ldots, \omega_{ZnL}$ are positive, negative, or zero numbers representing the $\boldsymbol{n_L}$ zeros.

Note that $F_L(s) \to 1$ as $s \to \infty$

The amplifier designer is usually particularly interested in the part of the low-frequency band that is close to the midband. Also, usually one of the poles — say, ω_{P1} — has a much higher frequency than all other poles. For frequencies ω close to the midband, $F_L(s)$ can be approximated by

$$F_L(s) \cong \frac{s}{(s+\omega_{P1})}$$

which is the transfer function of a first-order high-pass network. In this case the low-frequency response of the amplifier is ***dominated*** by the highest frequency pole and the lower 3-dB frequency is approximately equal to $\omega_L \cong \omega_{P1}$.

The better estimation for the **dominant-pole approximation** can be obtained from the following Equation:

$$\omega_L \cong \sqrt{\omega_{P1}^2 + \omega_{P2}^2 - 2\omega_{Z1}^2 - 2\omega_{Z2}^2}$$

EXAMPLE

The low-frequency response of an amplifier is characterized by the transfer function

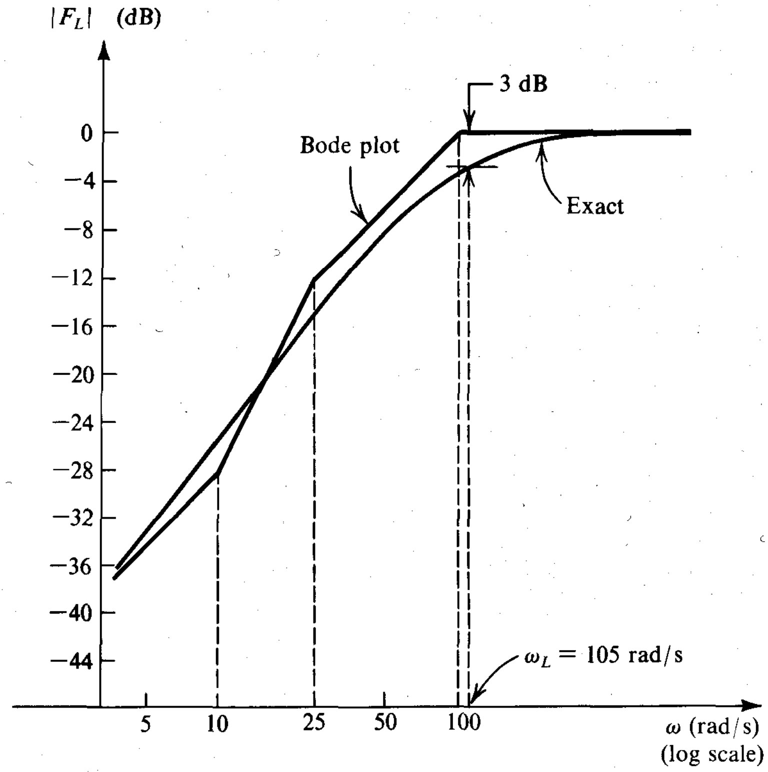
$$F_L(s) = \frac{s \cdot (s+10)}{(s+100) \cdot (s+25)}$$

Determine its 3-dB frequency, approximately and exactly.

Noting that the highest-frequency pole at 100 rad/s and ω_L =: 100 rad/s. A better estimate of ω_L can be obtained as follows:

$$\omega_L \cong \sqrt{100^2 + 25^2 - 2 - 2 \cdot 10^2} = 102 rad/s$$

The exact value of *WL* can be determined from the given transfer function as 105 rad/s.



Normalized low-frequency response of the amplifier in Example.

Note that this is a plot of the low-frequency response of the amplifier normalized relative to the midband gain. That is, if the midband gain is 100 dB, then the entire plot should be shifted upward by 100 dB.

The High-Frequency Response

The function $F_H(s)$ can be expressed in the general form

$$F_L(s) = \frac{(1+s/\omega_{Z1})\cdot(1+s/\omega_{Z2})\cdot\ldots\cdot(1+s/\omega_{ZnL})}{(1+s/\omega_{P1})\cdot(1+s/\omega_{P2})\cdot\ldots\cdot(1+s/\omega_{PnL})}$$ where

$\omega_{Z1}, \omega_{Z2}, \ldots, \omega_{ZnL}$ are positive numbers representing the frequencies of the $\boldsymbol{n_L}$ high-frequency poles

$\omega_{Z1}, \omega_{Z2}, \ldots, \omega_{ZnL}$ are positive, negative, or zero numbers representing the $\boldsymbol{n_L}$ high-frequency zeros.

Note that $F_L(s) \to 1$ as $s \to 0$

The amplifier designer is usually particularly interested in the part of the high-frequency band that is close to the midband. If one of the high-frequency poles—say, $\boldsymbol{\omega_{P1}}$ —is of much lower frequency than any of the other poles, then the high-frequency response of the amplifier will be *dominated* by this pole, and the function *Fn(s)* can be approximated by

$$F_L(s) \cong \frac{1}{(1+s/\omega_{P1})} \text{ and } \omega_H \cong \omega_{P1}$$

which is the transfer function of a first-order low-pass network.

If a dominant high-frequency pole does not exist or the better estimation the **dominant-pole approximation** the upper 3-dB frequency can be obtained from the following Equation:

$$\omega_H \cong \sqrt{1/\omega_{P1}^2 + 1/\omega_{P2}^2 + \ldots - 2/\omega_{Z1}^2 - 2/\omega_{Z2}^2 \ldots}$$

EXAMPLE

The high-frequency response of an amplifier is characterized by the transfer function

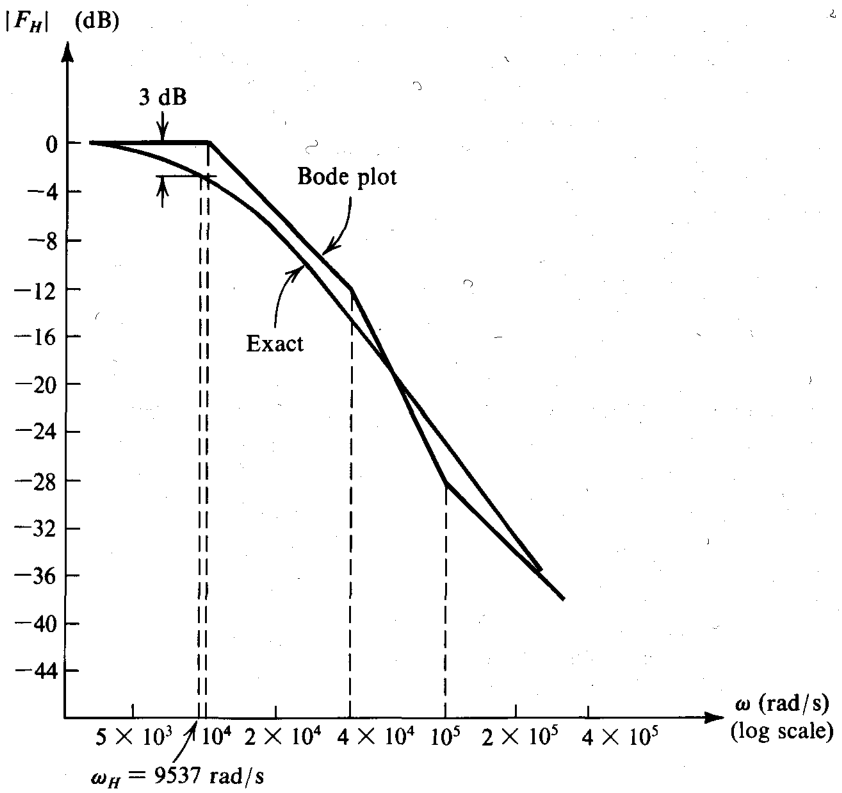
$$F_H(s) = \frac{1 - s/10^5}{\left(1 + s/10^4\right)\cdot\left(1 + s/4\cdot 10^4\right)}$$

Determine the 3-dB frequency approximately and exactly.

Solution

Noting that the lowest-frequency pole at 10^4 rad/s is two octaves lower than the second pole and a decade lower than the zero, we find that a dominant-pole situation almost exists and ω_H == 10^4 rad/s. A better estimate is as follows:

$$F_H(s) = \frac{1}{\sqrt{\dfrac{1}{10^8} + \dfrac{1}{16\cdot 10^8} - \dfrac{2}{10^{10}}}} = 9800 rad/s$$



THE FORMULAS FOR DETERMINING THE 3-dB FREQUENCIES OF AMPLIFIERS

Low-Frequency Response

$$A(s) \cong A_M F_L(s)$$

$$F_L(s) = \frac{(s + \omega_{Z1})(s + \omega_{Z2}) \cdots (s + \omega_{Zn_L})}{(s + \omega_{P1})(s + \omega_{P2}) \cdots (s + \omega_{Pn_L})}$$

If $\omega_{P1} \gg \omega_{P2}, \omega_{P3}, \cdots, \omega_{Z1}, \omega_{Z2}, \cdots$
then for frequencies near the midband:

$$F_L(s) \cong \frac{s}{s + \omega_{P1}} \qquad \text{(Dominant pole)}$$

and, $\omega_L \cong \omega_{P1}$.
Otherwise,

$$\omega_L \cong \sqrt{\omega_{P1}^2 + \omega_{P2}^2 + \cdots - 2(\omega_{Z1}^2 + \omega_{Z2}^2 + \cdots)}$$

High-Frequency Response

$$A(s) \cong A_M F_H(s)$$

$$F_H(s) = \frac{(1 + s/\omega_{Z1})(1 + s/\omega_{Z2}) \cdots (1 + s/\omega_{Zn_H})}{(1 + s/\omega_{P1})(1 + s/\omega_{P2}) \cdots (1 + s/\omega_{Pn_H})}$$

If $\omega_{P1} \ll \omega_{P2}, \omega_{P3}, \cdots, \omega_{Z1}, \omega_{Z2}, \cdots$
then for frequencies near the midband:

$$F_H(s) \cong \frac{1}{1 + s/\omega_{P1}} \qquad \text{(Dominant pole)}$$

and, $\omega_H \cong \omega_{P1}$.
Otherwise,

$$\omega_H = 1 \Bigg/ \sqrt{\frac{1}{\omega_{P1}^2} + \frac{1}{\omega_{P2}^2} + \cdots - 2\left(\frac{1}{\omega_{Z1}^2} + \frac{1}{\omega_{Z2}^2} + \cdots\right)}$$