Chapter 9

OPTIMUM DETECTION

9.1. INTRODUCTION

In this chapter we study the performance of digital communication systems in the presence of additive noise as measured by the probability of error and introduce the concept of the optimum signal detection. We assume throughout a distortionless channel, so the received signal is free of intersymbol interference (ISI). We also assume additive white gaussian noise (AWGN) with zero mean value, independent of the signal.

9.2. BINARY SIGNAL DETECTION AND HYPOTHESIS TESTING

Figure 9-1 portrays the operations of a binary receiver. The transmitted signal over a symbol interval $(0, T)$ is represented by

$$s_i(t) = \begin{cases} s_1(t) & 0 \leq t \leq T \quad \text{for } 1 \\ s_2(t) & 0 \leq t \leq T \quad \text{for } 0 \end{cases} \tag{9.1}$$

The received signal $r(t)$ by the receiver is represented by

$$r(t) = s_i(t) + n(t) \qquad i = 1, 2 \qquad 0 \leq t \leq T \tag{9.2}$$

where $n(t)$ is a zero-mean AWGN.

There are two separate steps involved in signal detection. The first step consists of reducing the received signal $r(t)$ to a single number $z(T)$. This operation can be performed by a linear filter followed by a sampler, as shown in block 1 of Fig. 9-1. The output of receiver (block 1), sampled at $t = T$, yields

$$z(T) = a_i(T) + n_o(T) \quad i = 1, 2 \tag{9.3a}$$

where $a_i(T)$ is the signal component of $z(T)$ and $n_o(T)$ is the noise component. We often write Eq. (9.3a) as

$$z = a_i + n_o \qquad i = 1, 2 \tag{9.3b}$$

Note that the noise component n_o is a zero-mean gaussian random variable, and thus z is a gaussian random variable with a mean of either a_1 or a_2 depending on whether $s_1(t)$ or $s_2(t)$ was sent. The sample z is sometimes called the *test statistic*.

The second step of the signal detection process consists of comparing the test statistic z to a threshold level λ in block 2 (threshold comparator) of Fig. 9-1. The final step in block 2 is to make the decision.

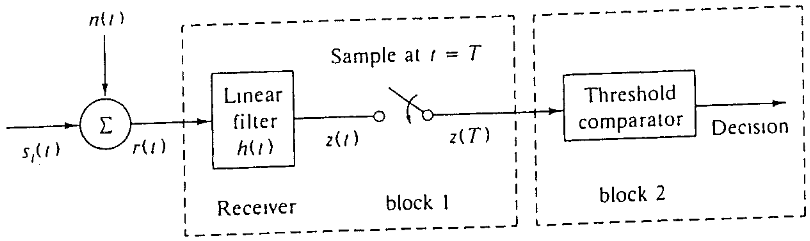

Fig. 9-1 Digital signal detection

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \lambda \tag{9.4}$$

where H_1 and H_2 are the possible hypotheses. Choosing H_1 is equivalent to deciding that signal $s_1(t)$ was sent, and choosing H_2 is equivalent to deciding that signal $s_2(t)$ was sent. Equation (9.4) indicates that hypothesis H_1 is chosen if $z > \lambda$, and hypothesis H_2 is chosen if $z < \lambda$. If $z = \lambda$, the decision can be an arbitrary one.

9.3 PROBABILITY OF ERROR AND MAXIMUM LIKELIHOOD DETECTOR

A. Probability of Error:

For the binary signal detection system, there are two ways in which errors can occur. That is, given that signal $s_1(t)$ was transmitted, an error results if hypothesis H_2 is chosen; or given that signal $s_2(t)$ was transmitted, an error results if hypothesis H_1 is chosen. Thus, the probability of error P_e is expressed as [Eq. (6.24)]

$$P_e = P(H_2|s_1)P(s_1) + P(H_1|s_2)P(s_2) \tag{9.5}$$

where $P(s_1)$ and $P(s_2)$ are the a priori probabilities that $s_1(t)$ and $s_2(t)$, respectively are transmitted.

When symbols 1 and 0 occur with equal probability, that is, $P(s_1) = P(s_2) = \frac{1}{2}$,

$$P_e = \tfrac{1}{2}[P(H_2|s_1) + P(H_1|s_2)] \tag{9.6}$$

B. Maximum Likelihood Detector:

A popular criterion for choosing the threshold λ of Eq. (9.4) is based on minimizing the probability of error of Eq. (9.5). The computation for this minimum error value of $\lambda = \lambda_0$ starts with forming the following *likelihood ratio test* (Prob. 9.1)

$$\Lambda(z) = \frac{f(z|s_1)}{f(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)} \tag{9.7}$$

where $f(z|s_i)$ is the conditional pdf known as the *likelihood* of s_i. The ratio $\Lambda(z)$ is known as the *likelihood ratio*. Equation (9.7) states that we should choose hypothesis H_1 if the likelihood ratio $\Lambda(z)$ is greater than the ratio of a priori probabilities. If $P(s_1) = P(s_2)$, Eq. (9.7) reduces to

$$\Lambda(z) = \frac{f(z|s_1)}{f(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} 1 \tag{9.8a}$$

or

$$f(z|s_1) \underset{H_2}{\overset{H_1}{\gtrless}} f(z|s_2) \tag{9.8b}$$

If $P(s_1) = P(s_2)$ and the likelihoods $f(z|s_i)$ $(i = 1, 2)$ are symmetric, then Eq. (9.7) yields the criterion (Prob. 9.2)

$$z \underset{H_2}{\overset{H_1}{\gtrless}} \lambda_0 \tag{9.9}$$

where

$$\lambda_0 = \frac{a_1 + a_2}{2} \tag{9.10}$$

It can be shown that the threshold λ_0 represented by Eq. (9.10) is the *optimum threshold* for minimizing the error of probability (Prob. 9.3). The criterion of Eq. (9.9) is known as the *minimum error criterion*. A detector that minimizes the error probability (for the case where the signal classes are equally likely) is also known as a *maximum likelihood detector*

C. Probability of Error with Gaussian Noise:

The pdf of the gaussian random noise n_o in Eq. (9.3b) is [Eq. (6.91)]

$$f_{n_o}(\xi) = \frac{1}{\sqrt{2\pi}\sigma_{n_o}} e^{-\xi^2/(2\sigma_{n_o}^2)} \tag{9.11}$$

where $\sigma_{n_o}^2$ is the noise variance. It follows from Eqs. (9.3b) and (9.11) that

$$f(z|s_1) = \frac{1}{\sqrt{2\pi}\sigma_{n_o}} e^{-(z-a_1)^2/(2\sigma_{n_o}^2)} \tag{9.12a}$$

$$f(z|s_2) = \frac{1}{\sqrt{2\pi}\sigma_{n_o}} e^{-(z-a_2)^2/(2\sigma_{n_o}^2)} \tag{9.12b}$$

which are illustrated in Fig. 9-2.

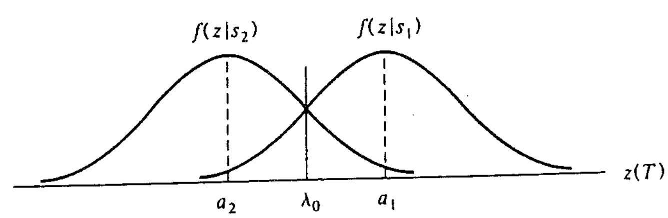

Fig. 9-2 Conditional pdf

Now

$$P(H_2|s_1) = \int_{-\infty}^{\lambda_0} f(z|s_1)\, dz \tag{9.13a}$$

$$P(H_1|s_2) = \int_{\lambda_0}^{\infty} f(z|s_2)\, dz \tag{9.13b}$$

Because of the symmetry of $f(z|s_i)$, Eq. (9.6) reduces to

$$P_e = P(H_2|s_1) = P(H_1|s_2) \tag{9.14}$$

Thus, the probability of error P_e is numerically equal to the area under the "tail" of either likelihood function $f(z|s_1)$ or $f(z|s_2)$ falling on the "incorrect" side of the threshold;

$$P_e = \int_{\lambda_0}^{\infty} f(z|s_2)\, dz \tag{9.15}$$

where $\lambda_0 = (a_1 + a_2)/2$ is the optimum threshold [Eq. (*9.10*)]. Using Eq. (*9.12b*), we have

$$P_e = \int_{\lambda_0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{n_o}} e^{-(z-a_2)^2/(2\sigma_{n_o}^2)} dz$$

Let $y = (z - a_2)/\sigma_{n_o}$ Then $\sigma_{n_o} dy = dz$ and

$$P_e = \int_{(a_1-a_2)/(2\sigma_{n_o})}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy = Q\left(\frac{a_1 - a_2}{2\sigma_{n_o}}\right) \tag{9.16}$$

where $Q(\,)$ is the complementary error function, or the Q function defined in Eq. (*6.93*). The values of the Q function are tabulated in App. C.

9.4 OPTIMUM DETECTION

In this section we consider optimizing the linear filter in the receiver (block 1) of Fig. 9-1 by minimizing the probability of error P_e.

A. The Matched Filter:

A matched filter is a linear filter designed to provide the maximum output SNR for a given transmitted signal. Consider that a known signal $s(t)$ plus AWGN $n(t)$ is the input to an LTI filter followed by a sampler, as shown in Fig. 9-1. Let $a(t)$ be the output of the filter. Then from Eq. (*9.3a*), at $t=T$ we have

$$\left(\frac{S}{N}\right)_o = \frac{a^2(T)}{E[n_o^2(T)]} = \frac{a^2(T)}{\sigma_{n_o}^2} \tag{9.17}$$

We wish to find the filter frequency response $H_0(\omega)$ that maximizes Eq. (*9.17*). It can be shown that (Prob. 9 7)

$$\left(\frac{S}{N}\right)_o \leq \frac{2}{\eta}\frac{1}{2\pi}\int_{-\infty}^{\infty} |S(\omega)|^2\, d\omega = \frac{2E}{\eta} \tag{9.18}$$

where $S(\omega) = \mathscr{F}[s(t)]$, $\eta/2$ is the power spectral density of the input noise, and E is the energy of the input signal $s(t)$. Note that the right-hand side of this inequality does not depend on $H(\omega)$ but only on the input signal energy and the power spectral density of noise. Thus,

$$\left(\frac{S}{N}\right)_{o_{\max}} = \frac{2E}{\eta} \tag{9.19}$$

The equality in Eq. (*9.18*) holds only if the optimum filter frequency response $H_0(\omega)$ is employed such that (Prob. 9 7)

$$H(\omega) = H_0(\omega) = S^*(\omega)e^{-j\omega T} \tag{9.20}$$

where * denotes the complex conjugate.

The impulse response $h(t)$ of this optimum filter is [see Eqs. (*1.18*) and (*1.21*)]

$$h(t) = \mathscr{F}^{-1}[H(\omega)] = \begin{cases} s(T-t) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \tag{9.21}$$

Equation (*9.21*) and Fig. 9-3 illustrate the matched filter's basic property: The impulse response of the matched filter is a delayed version of the mirror image of the signal form.

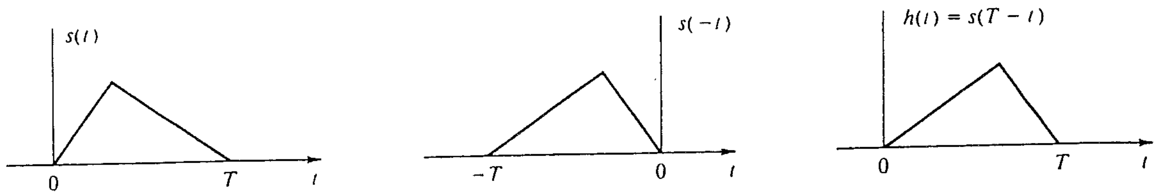

Fig. 9-3 Matched filter characteristics

B. Correlator:

The output $z(t)$ of a causal filter can be expressed as [Eq. (*2.8*)]

$$z(t) = r(t) * h(t) = \int_0^t r(\tau)h(t-\tau)\, d\tau \tag{9.22}$$

Substituting $h(t)$ of Eq. (*9.21*) into Eq. (*9.22*), we obtain

$$z(t) = \int_0^t r(\tau)s[T-(t-\tau)]\, d\tau \tag{9.23}$$

When $t = T$, we have

$$z(T) = \int_0^T r(\tau)s(\tau)\, d\tau \tag{9.24}$$

The operation of Eq. (*9.24*) is known as the *correlation* of $r(t)$ and $s(t)$.

Since the matched filter output and the correlator output are identical at the sampling time $t = T$ the matched filter and correlator depicted in Fig. 9-4 are used interchangeably

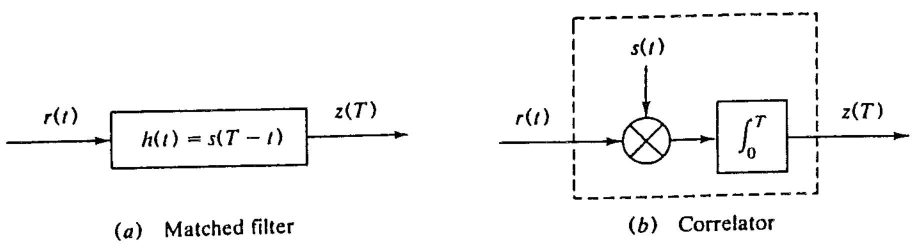

Fig. 9-4 Equivalence of matched filter and correlator

C. Optimum Detection:

To minimize P_e of Eq. (*9.16*), we need to determine the linear filter that maximizes $(a_1 - a_2)/(2\sigma_{n_o})$ or, equivalently, that maximizes

$$\frac{(a_1 - a_2)^2}{\sigma_{n_o}^2} \tag{9.25}$$

where $a_1 - a_2$ is the difference of the signal components at the filter output, at time $t = T$, hence, $(a_1 - a_2)^2$ is the instantaneous power of the difference signal, and $\sigma_{n_o}^2$ is the average output noise power.

Consider a filter that is matched to the input signal $s_1(t) - s_2(t)$. From Eqs. (*9.17*) and (*9.19*), we have

$$\left(\frac{S}{N}\right)_o = \frac{(a_1 - a_2)^2}{\sigma_{n_o}^2} = \frac{E_d}{\eta/2} = \frac{2E_d}{\eta} \tag{9.26}$$

where $\eta/2$ is the power spectral density of the noise at the filter input and E_d is the energy of the difference signal at the filter input:

$$E_d = \int_0^T [s_1(t) - s_2(t)]^2\, dt \tag{9.27}$$

Hence, using Eqs. (*9.16*) and (*9.26*), we obtain

$$P_e = Q\left(\frac{a_1 - a_2}{2\sigma_{n_o}}\right) = Q\left(\sqrt{\frac{E_d}{2\eta}}\right) \tag{9.28}$$

9.5 ERROR PROBABILITY PERFORMANCE OF BINARY TRANSMISSION SYSTEMS

By using Eq. (*9.28*), the probabilities of error for various binary transmission systems are given in the following.

A. Unipolar Baseband Signaling:

$$s_i(t) = \begin{cases} s_1(t) = A & 0 \leqslant t \leqslant T \\ s_2(t) = 0 & 0 \leqslant t \leqslant T \end{cases} \tag{9.29}$$

The probability of error P_e is

$$P_e = Q\left(\sqrt{\frac{A^2T}{2\eta}}\right) = Q\left(\sqrt{\frac{E_b}{\eta}}\right) \tag{9.30}$$

where $E_b = A^2T/2$ is the average signal energy per bit.

B. Bipolar Baseband Signaling:

$$s_i(t) = \begin{cases} s_1(t) = +A & 0 \leqslant t \leqslant T \\ s_2(t) = -A & 0 \leqslant t \leqslant T \end{cases} \tag{9.31}$$

The probability of error P_e is (Prob. 9.12)

$$P_e = Q\left(\sqrt{\frac{2A^2T}{\eta}}\right) = Q\left(\sqrt{\frac{2E_b}{\eta}}\right) \tag{9.32}$$

where $E_b = A^2T$ is the average signal energy per bit.

C. Amplitude-Shift Keying (or On-Off Keying):

$$s_i(t) = \begin{cases} s_1(t) = A\cos\omega_c t & 0 \leqslant t \leqslant T \\ s_2(t) = 0 & 0 \leqslant t \leqslant T \end{cases} \tag{9.33}$$

with T an integer times $1/f_c$. The probability of error P_e is

$$P_e = Q\left(\sqrt{\frac{A^2T}{4\eta}}\right) = Q\left(\sqrt{\frac{E_b}{\eta}}\right) \tag{9.34}$$

where $E_b = A^2T/4$ is the average signal energy per bit.

D. Phase-Shift Keying:

$$s_i(t) = \begin{cases} s_1(t) = A\cos\omega_c t & 0 \le t \le T \\ s_2(t) = A\cos(\omega_c t + \pi) & \\ \quad\;\; = -A\cos\omega_c t & 0 \le t \le T \end{cases} \tag{9.35}$$

with T an integer times $1/f_c$. The probability of error P_e is (Prob. 9.14)

$$P_e = Q\left(\sqrt{\frac{A^2T}{\eta}}\right) = Q\left(\sqrt{\frac{2E_b}{\eta}}\right) \tag{9.36}$$

where $E_b = A^2T/2$ is the average signal energy per bit.

E. Frequency-Shift Keying:

$$s_i(t) = \begin{cases} s_1(t) = A\cos\omega_1 t & 0 \le t \le T \\ s_2(t) = A\cos\omega_2 t & 0 \le t \le T \end{cases} \tag{9.37}$$

If we assume $\omega_1 T \gg 1$, $\omega_2 T \gg 1$, and $(\omega_1 - \omega_2)T \gg 1$, then the probability of error P_e is (Prob. 9.17)

$$P_e \approx Q\left(\sqrt{\frac{A^2T}{2\eta}}\right) = Q\left(\sqrt{\frac{E_b}{\eta}}\right) \tag{9.38}$$

where $E_b = A^2T/2$ is the average signal energy per bit.

Solved Problems

PROBABILITY OF ERROR AND MAXIMUM LIKELIHOOD DETECTOR

9.1. Derive the likelihood ratio test given by Eq. (9.7), that is,

$$\Lambda(z) = \frac{f(z|s_1)}{(z|s_2)} \underset{H_2}{\overset{H_1}{\gtrless}} \frac{P(s_2)}{P(s_1)}$$

A reasonable receiver decision rule is to choose hypothesis H_1 if the a posteriori probability $P(s_1|z)$ is greater than the a posteriori probability $P(s_2|z)$. Otherwise, we should choose hypothesis H_2 (See Prob. 6.15)

Hence,

$$P(s_1|z) \underset{H_2}{\overset{H_1}{\gtrless}} P(s_2|z) \tag{9.39}$$

or

$$\frac{P(s_1|z)}{P(s_2|z)} \underset{H_2}{\overset{H_1}{\gtrless}} 1 \tag{9.40}$$

The decision criterion of Eq. (9.40) is called the *maximum a posteriori* (MAP) criterion.