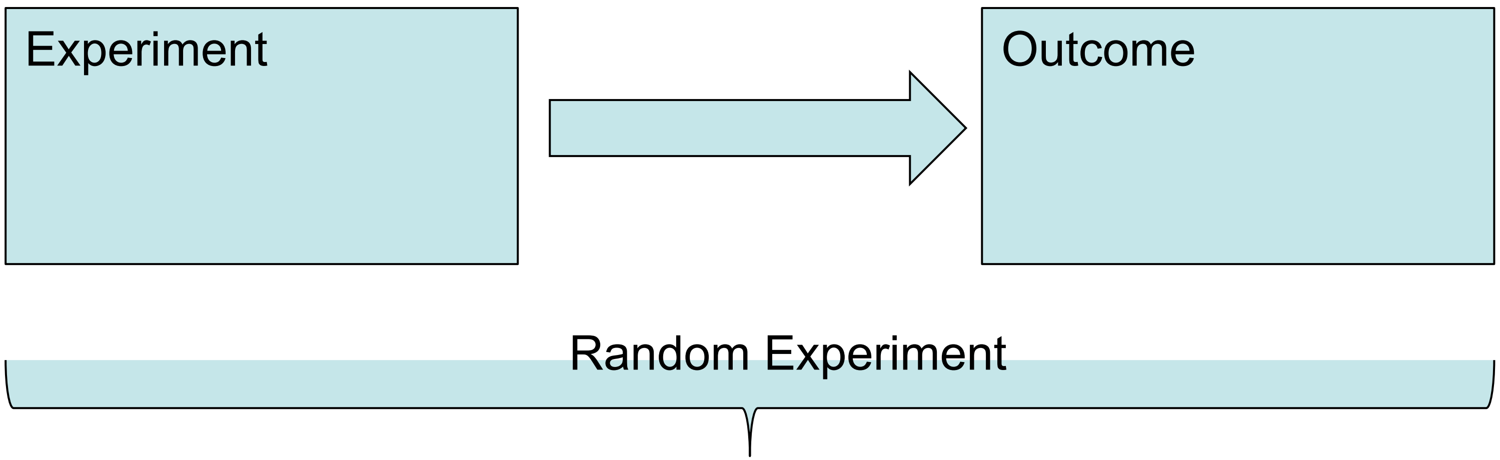
Chapter 6

Probability and Random Processes

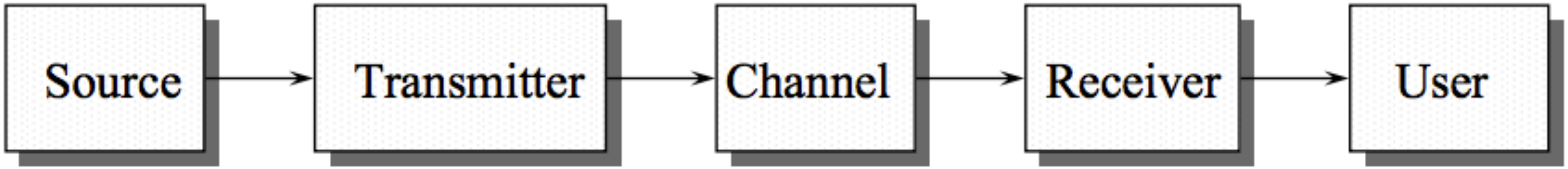
Random Experiment

- The fundamental concept in probability theory is the concept of **random experiment**, which is any experiment whose outcome cannot be predicted with certainty

- A simple example is coin tossing experiment. We know that heads and tails are possible outcomes, although the outcome (head or tail?) of a particular experiment (toss) is uncertain

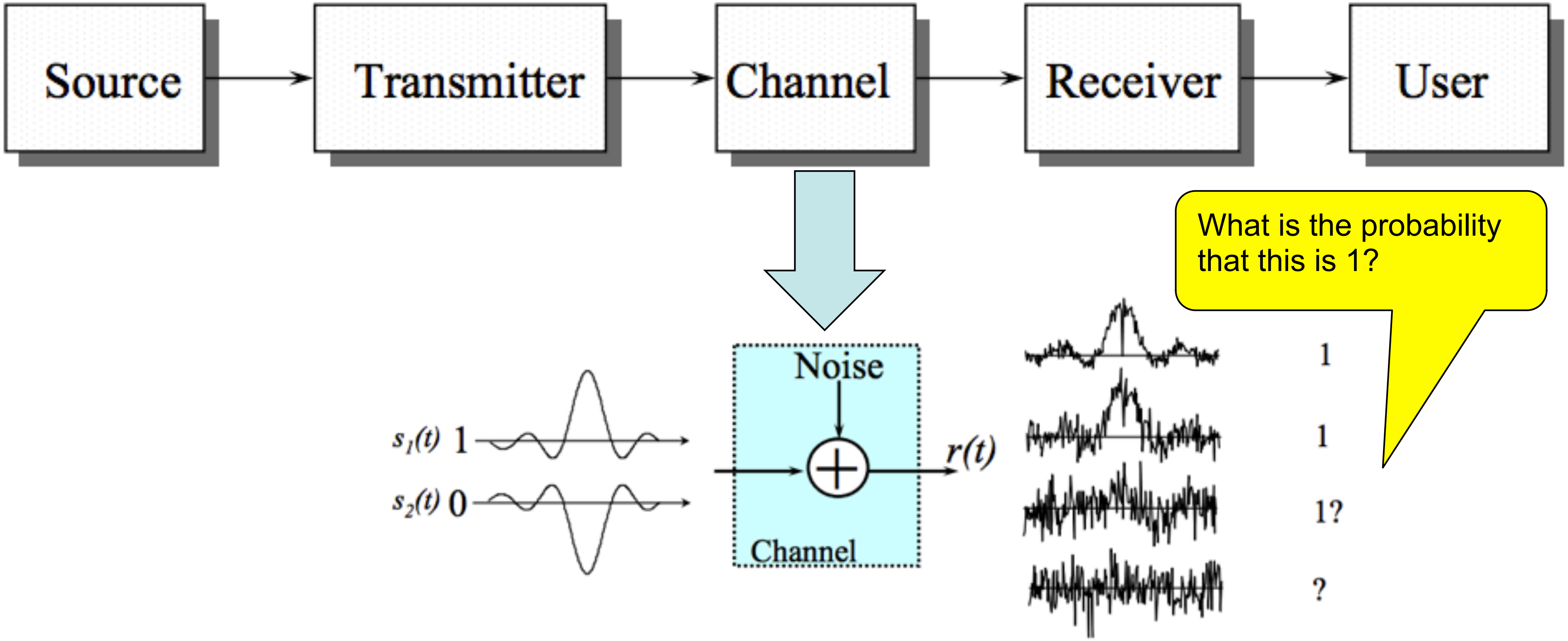


A General Communication System

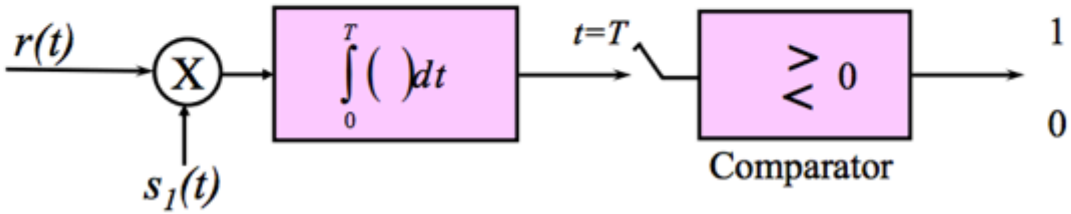


- ***Source***: Speech, Video, etc.
- ***Transmitter***: Conveys information
- ***Channel***: Invariably distorts signals
- ***Receiver***: Extracts information signal
- ***User***: Utilizes information

Why Learn about Probability Theory?



Optimum (Correlation) Receiver:



Back Probability Concepts

- Let us define the following concepts associated with a random experiment:
  - **Outcome** (ξ) – the result of a random experiment
  - **Sample space** (Ω) – the set of all possible outcomes of a random experiment
  - **Event** (A) – any collection of outcomes, in other words, a subset of Ω
  - The empty subset ϕ, is called the **null** or **impossible event**, and the whole set Ω is called the **whole** or **sure event**

- Example: Roll a dice
  - Outcomes: landing with a 1, 2, 3, 4, 5, or 6 face up.
  - Sample Space: S ={1, 2, 3, 4, 5, 6}
  - Event: outcome is larger than 4
  - Frequency of 1 happening = 10/60 = 1/6 (10 occurrence; 60 trials)
  - We obtain Probability or Likelihood → We try INFINIT times!

Probability Axioms (P1-P3)

- In the axiomatic approach, the probability is defined as a function that assigns a real number, denoted by $P(A)$, to every event A in the sample space Ω such that:

P1 $0 \le P(A) \le 1$

P2 The whole event Ω will occur each time we perform the random experiment

$$P(\Omega) = 1$$

P3 If the events are **mutually exclusive** (i.e., can not occur at the same time), the probability of their union is the sum of their probabilities

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

Probability Axioms

Using P1-P3:

- By using the above axioms, we can derive following important properties of the probability function:

P4 The probability of the null event is zero.

$$P(\phi)=0$$

P5 $P(\bar{A})=1-P(A)$, $\bar{A}$ = complement of A

- If the events $A_1, A_2, \ldots$ are not mutually exclusive, the probability of their union is upper-bounded by the sum of probabilities of the constituent events. That is,

$$P(A_1 \cup A_2 \cup \ldots\ldots) \leq P(A_1)+P(A_2)+\ldots\ldots$$ **Union Bound**

Example

- Rolling a dice. S = {1, 2, 3, 4, 5, 6}
- Find intersection and union of two events A and B

  – Defining Events: Let A = {1, 2, 3} and B = {1, 3, 5}

  – Union of sets: AUB = {1,2,3,5}

  – Intersection: $A \bigcap B = \{1,3\}$

  – A′ = {4,5,6}

Example of Union and Intersection

- A card is drawn from a well-shuffled deck of 52 playing cards. What is the probability that it is a queen or a heart?

$$Q = \text{Queen} \quad \text{and} \quad H = \text{Heart}$$

$$P(Q) = \frac{4}{52},\ P(H) = \frac{13}{52},\ P(Q \cap H) = \frac{1}{52}$$

$$P(Q \cup H) = P(Q) + P(H) - P(Q \cap H)$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} \quad = \frac{16}{52} = \frac{4}{13}$$

Conditional Probability

- The probability $P(A)$ is *a priori* probability of the occurrence of an event A
  - Reflects our knowledge of A before the random experiment takes place
- The **conditional probability** $P(A|B)$ is the *a posteriori* probability of event A knowing that event B has already occurred
- It is defined as

Note: We are assuming A and B are not independent!

$$P(A \mid B) = \frac{P(AB)}{P(B)},$$

provided $P(B) > 0$

- Conditioning by event B has the effect of restricting the universe of outcomes for the event A to the subset B of Ω

a-pre-ori / a-poste-rio-rayh

Independent Events

- A and B are said to be independent events if

$$P(AB) = P(A)P(B)$$

- One should not confuse independent events with mutually exclusive or disjoint events
  - Mutually exclusive events have no outcome in common, i.e., $AB = \phi$ implying that $P(AB) = 0$
  - Independent events in most cases are not disjoint
- Substituting into the definition of conditional probability yields

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

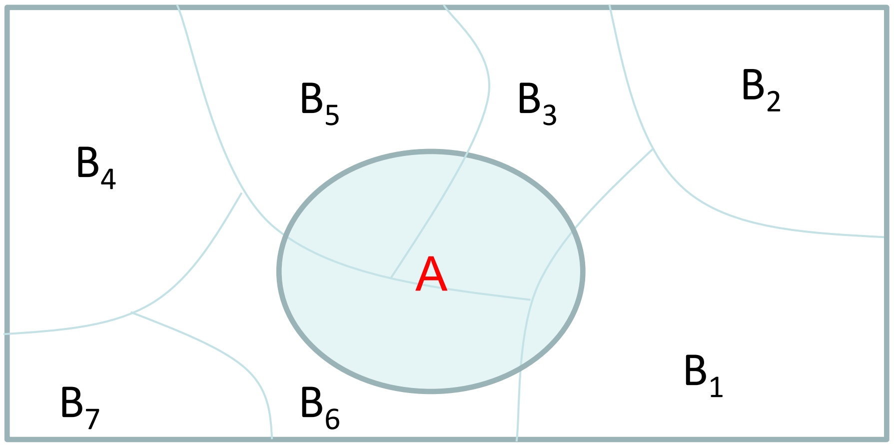
- $\Rightarrow$ that the occurrence of B does not provide any more information about the event A

Example

- 1A & 1B
- 1C

Rule (Law) of Total Probability

Basically: we can calculate the probability of an event based on other events



$$p(A) = \sum P(B_i) P(A \mid B_i)$$

Bayes' Theorem (simple version)

Theorem (Bayes' Theorem)

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

This lets us express the probability of A given B, in terms of the probability of B given A.

Alternate formulation of Bayes' Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

where we used

$$P(B) = P(B \cap A) + P(B \cap A^c) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

http://www.math.ucsd.edu/~gptesler/186/slides/bayesthm_14-handout.pdf

Full version of Bayes' Theorem

Definition (Partition of S)

Events $A_1, \ldots, A_n$ *partition* the sample space S when

- $S = A_1 \cup \cdots \cup A_n$.
- $A_i \cap A_j = \emptyset$ for $i \neq j$. *(pairwise mutually exclusive)*
- $P(A_i) > 0$ for all i.

In other words, $A_1, \ldots, A_n$ are all nonempty with positive probability, and every element of the sample space is in exactly one of them.

Theorem (Bayes' Theorem)

Let $A_1, \ldots, A_n$ be mutually exclusive events that partition sample space S, and B be any event on S. Then

- $P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$
- *If $P(B) > 0$ then for each $j = 1, \ldots, n$,*

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

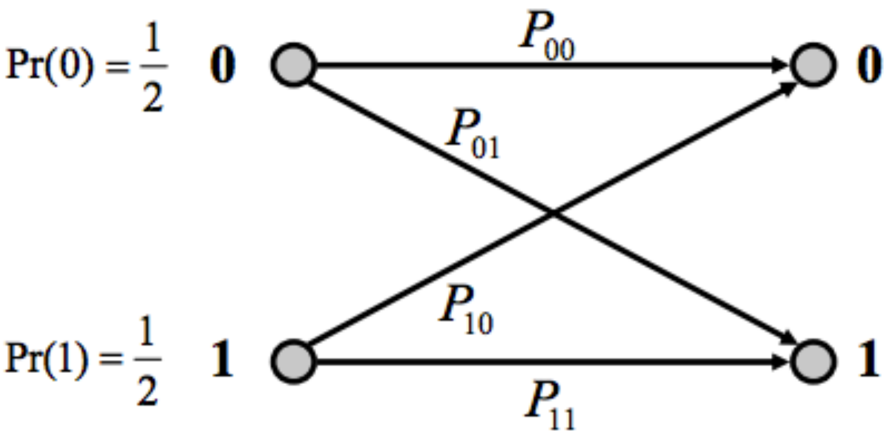
Can you prove this?

http://www.math.ucsd.edu/~gptesler/186/slides/bayesthm_14-handout.pdf

Example

- 1D
- 1E

Example of Conditional Probability



P_{00}=P[receive 0 | 0 sent]

P_{10}=P[receive 0 | 1 sent]

P_{01}=P[receive 1 | 0 sent]

P_{11}=P[receive 1 | 1 sent]

Given:

$$P_{01} = 0.01 \quad \Rightarrow \quad P_{00} = 1 - P_{01} = 0.99$$

$$P_{10} = 0.01 \quad \Rightarrow \quad P_{11} = 1 - P_{10} = 0.99$$

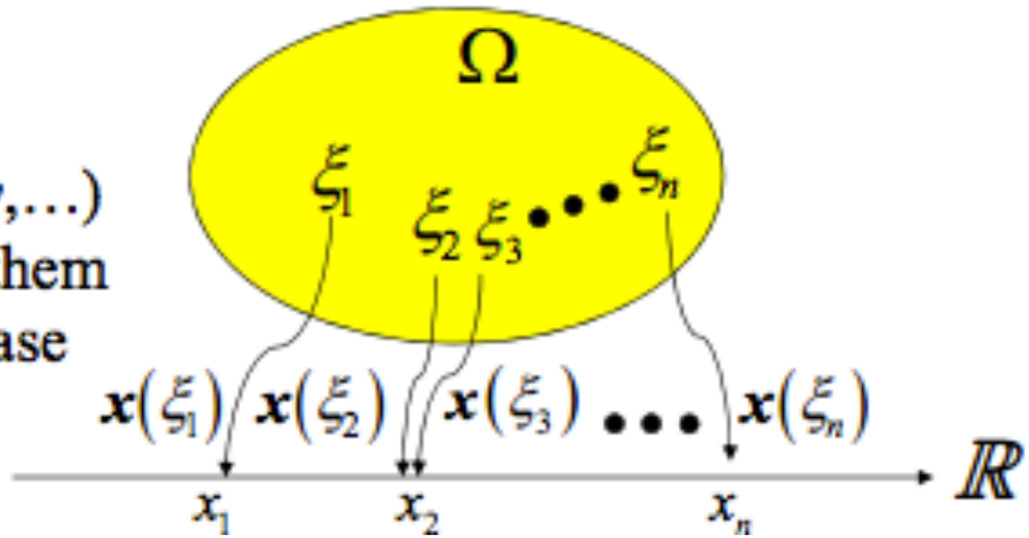
$$\Pr(e) = \Pr(0)\cdot P_{01} + \Pr(1)\cdot P_{10} = \frac{1}{2}\cdot 0.01 + \frac{1}{2}\cdot 0.01$$
$$= 0.01$$

http://www.ece.tamu.edu/~georghiades/courses/ftp455/intro.pdf

Random Variable

- A **random variable** is defined as a rule that assigns a real number to each possible outcome $\xi \in \Omega$ of a random experiment
  - Thus, random variable is a function that maps every outcome $\xi \in \Omega$ to a real number x as illustrated in Figure

Conceptual model of a random variable



We will denote random variables in a bold font ($\boldsymbol{x}, \boldsymbol{y}, \ldots$) and the values assumed by them are displayed by the lowercase letters ($x, y, \ldots$).

Discrete Random Variables

- Random variables may be discrete, continuous or mixed depending upon the range of values they assume
- A **discrete** random variable $\boldsymbol{x}$ can take on a **countable** number of values $x_1, x_2, x_3, \ldots$ with probabilities

$$P\{\boldsymbol{x} = x_i\}, i = 0, 1, 2, \ldots$$

  - e.g., # of defective chips from a semiconductor wafer
- A **probability mass function (PMF)** $p_x(x_i)$ completely characterizes a discrete random variable. It is defined as

$$p_x(x_i) = P\{\boldsymbol{x} = x_i\}$$

- Since $p_x(x_i)$ is a probability, it satisfies following properties
$0 \leq p_x(x_i) \leq 1, \qquad \sum_i p_x(x_i) = \sum_i P\{\boldsymbol{x}(\xi_i) = x_i \mid \xi_i \in \Omega\} = 1$

Continuous Random Variables

- A continuous random variable x takes values in a continuous set of numbers. The range of x may include the whole real line or an interval thereof
- Continuous random variables model many real life phenomena that include file download time on Internet, voltage across a resistor, and phase of a carrier signal produced by a radio transmitter
- Therefore, we can not use the PMF for a continuous random variable. Instead we shall use the cumulative distribution function which serves as an appropriate probability measure for any random variable

Example

- See notes DD1

Cumulative Distribution Function (CDF)

- The **cumulative distribution function (CDF)**, $F_x(x)$, of a random variable x is defined as

  $$F_x(x) = P\{x \le x\}$$

- For any real number x, the CDF measures the probability that the random variable x is no larger than x
  - (a) $0 \le F_x(x) \le 1$
  - (b) $\lim_{x \to -\infty} F_x(x) = 0$ and $\lim_{x \to \infty} F_x(x) = 1$
  - (c) $P\{a < x \le b\} = F_x(b) - F_x(a)$
  - (d) $F_x(x)$ is nondecreasing

Density Function

- A probability density function (PDF), $f_x(x)$, of a continuous random variable x is derivative of its CDF. That is.

$$f_x(x) = \frac{dF_x(x)}{dx}$$

Density Function

Distribution Function

- The CDF of a continuous random variable x is integral of its PDF

$$F_x(a) = \int_{-\infty}^{a} f_x(x)dx$$

→PDF is a continuous random variable is a function which can be integrated to obtain the probability that the random variable takes a value in a given interval.

- (a) $f_x(x) \geq 0$
- (b) $\int_{-\infty}^{\infty} f_x(x)dx = 1$
- (c) $\int_{a}^{b} f_x(x)dx = P\{a < x \leq b\}$

Example

- CC1- See notes

The PDF of a random variable is given by

$$f_x(x) = \begin{cases} Ce^{-x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Find

a. The constant C
b. The CDF $F_x(x)$
c. $P\{0 < x \leq 5\}$
d. $P\{-3 < x \leq 3\}$

Common **Discrete** RVs

- Uniform
- Bernoulli
- Binomial
- Poisson

Uniform RV

- Totally Random – Equally likely events:

$$P\{\mathbf{x} = k\} = \frac{1}{M}, \qquad k = 0, 1, 2, \ldots, M-1$$

Its PMF can be

$$p_{\mathbf{x}}(x) = \begin{cases} 1/M, & k = 0, 1, 2, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases}$$

Bernoulli Random Variable

- Binary Random variable where $0 < p < 1$
- Bernoulli random variables are used to model random experiments whose outcomes are binary
  - For example, whether a bit is received in error, or whether a packet is dropped by a congested router

$$P\{\mathbf{x} = 1\} = p$$

$$P\{\mathbf{x} = 0\} = 1 - p$$

Its PMF can be written

$$p_{\mathbf{x}}(x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \end{cases}$$

Binomial Random Variable

- Binomial random variables model the number of successes in a sequence of n independent trials of a random experiment, each of which yields success with probability p.
- x RV is a binomial random variable if its PMF is of the form

$$p_x(k) = P\{x = k\} = P\{k \text{ success in } n \text{ trials}\}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n$$

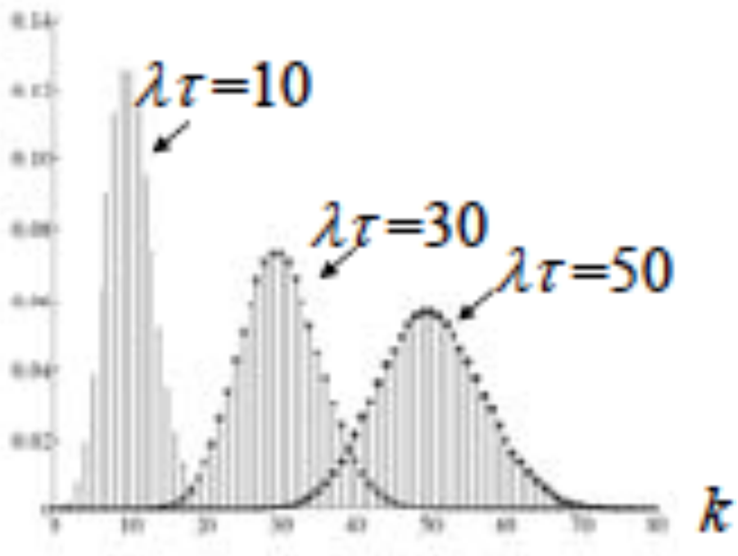
Remember: Combination Example: Picking a team of
3 people from a group of 10. C(10,3) = 10!/(7! * 3!)

Poisson Random Variable

- The Poisson random variable x models the number of events (k) occurring in any interval $(t_o, t_o+\tau)$ if the occurrence of these events, at an average rate λ, is independent of t_o and depends only on the length of interval τ
- It is common in the literature to refer to the occurrence of a Poisson event as an arrival
- x is a Poisson random variable if its PMF is of the form

$$p_x(k) = P(x=k) = P\{k \text{ arrivals in interval } \tau\} = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}, \quad k = 0,1,2,\cdots,\infty$$

where λ = average arrival rate



Examples

- AA1
- BB1

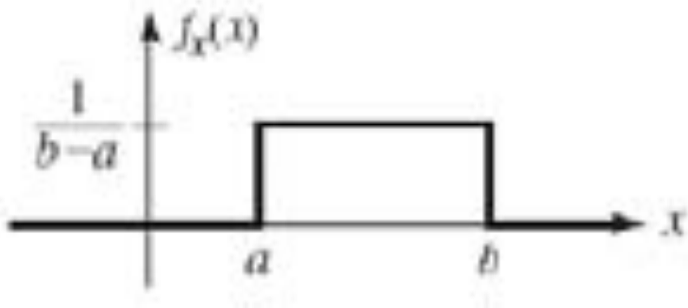
Common **Continuous** Random Variables

- Here we introduce three important continuous random variables:
  - Uniform
  - Gaussian
  - Exponential
  - Poisson
  - Rayleigh

Uniform Random Variable

- x is a uniform random variable if its PDF is given by

$$f_x(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

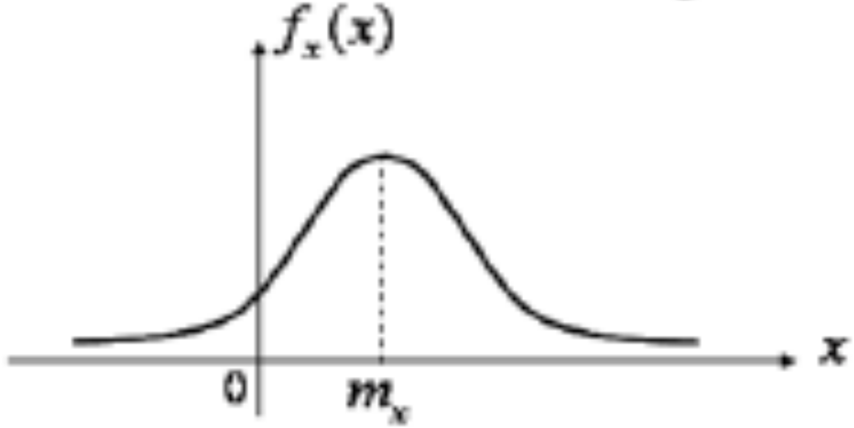


- The uniform random variable is a good model when each outcome of a random experiment is equally likely, and constrained to lie in the interval [b, a], $b > a$.

Gaussian or Normal Random Variable

- x is a normal or Gaussian random variable if its PDF is given by

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(x-m_x)^2/2\sigma_x^2}$$



- Characterized by **mean m_x and variance σ_x^2**
  - **σ_x called the standard deviation**
- A Gaussian random variable with mean m_x and variance σ_x^2 is denoted by $\mathcal{N}(m_x, \sigma_x^2)$
- It is most frequently used random variable in the analysis and modeling of communication systems.

Gaussian or Normal Random Variable (contd)

- The CDF $F_x(x)$ of the Gaussian random variable x is given by

$$F_x(x) = P\{x \le x\} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(t-m_x)^2/2\sigma_x^2} dt$$

- There is no closed form solution for the integral on the right hand side. However, it can be written in terms of the Q-function as

$$F_x(x) = 1 - Q\left(\frac{x - m_x}{\sigma_x}\right) = Q\left(\frac{m_x - x}{\sigma_x}\right)$$

Standard Deviation

where

$$Q(a) = P\{x > a\} = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-y^2/2} dy$$

Using Q-Function table
Q(a) can be found!
→Next

Gaussian or Normal Random Variable (properties)

- Remember:
  - Q-Function is the area under standard normal RV

$$Q(0) = \frac{1}{2}$$

- Important Properties:

$$Q(-\infty) = 1$$

$$Q(-x) = \frac{1}{\sqrt{2\pi}}\int_{-x}^{\infty} e^{-y^2/2}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy - \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-y^2/2}dy = 1 - Q(x)$$

$$\Rightarrow \quad 1 - Q\left(\frac{x - m_x}{\sigma_x}\right) = Q\left(\frac{m_x - x}{\sigma_x}\right)$$

- Integrals for Q(z cannot be evaluated in closed form. However, for large values of z, very good closed-form approximations can be obtained, and for small values of z, numerical integration techniques can be applied easily

Approx. Upp. Bound

$$Q(z) = \int_z^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2} d\lambda = \int_z^{\infty} u\,dv = uv\Big|_z^{\infty} - \int_z^{\infty} v\,du \quad < \frac{1}{\sqrt{2\pi}\,z} e^{-z^2/2}, \quad z > 0$$

Table of Q-Function

Table 1: Values of $Q(x)$ for $0 \leq x \leq 9$

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0.00	0.5	2.30	0.010724	4.55	2.6823×10^{-6}	6.80	5.231×10^{-12}
0.05	0.48006	2.35	0.0093867	4.60	2.1125×10^{-6}	[illegible]	3.6925×10^{-12}
0.10	0.46017	2.40	0.0081975	4.65	1.6597×10^{-6}	6.90	[illegible]
0.15	0.44038	2.45	0.0071428	4.70	1.3008×10^{-6}	6.95	[illegible]
0.20	0.42074	2.50	0.0062097	4.75	1.0171×10^{-6}	7.00	[illegible]
0.25	0.40129	2.55	0.0053861	4.80	7.9333×10^{-7}	7.05	8.9459×10 [illegible]
0.30	0.38209	2.60	0.0046612	4.85	6.1731×10^{-7}	7.10	6.2378×10^{-13}
0.35	0.36317	2.65	0.0040246	4.90	4.7918×10^{-7}	7.15	4.3389×10^{-13}
0.40	0.34458	2.70	0.003467	4.95	3.7107×10^{-7}	7.20	3.0106×10^{-13}
0.45	0.32636	2.75	0.0029798	5.00	2.8665×10^{-7}	7.25	2.0839×10^{-13}
0.50	0.30854	2.80	0.0025551	5.05	2.2091×10^{-7}	7.30	1.4388×10^{-13}
0.55	0.29116	2.85	0.002186	5.10	1.6983×10^{-7}	7.35	9.9103×10^{-14}
0.60	0.27425	2.90	0.0018658	5.15	1.3024×10^{-7}	7.40	6.8092×10^{-14}
0.65	0.25785	2.95	0.0015889	5.20	9.9644×10^{-8}	7.45	4.667×10^{-14}
0.70	0.24196	3.00	0.0013499	5.25	7.605×10^{-8}	7.50	3.1909×10^{-14}
0.75	0.22663	3.05	0.0011442	5.30	5.7901×10^{-8}	7.55	2.1763×10^{-14}
0.80	0.21186	3.10	0.0009676	5.35	4.3977×10^{-8}	7.60	1.4807×10^{-14}
0.85	0.19766	3.15	0.00081635	5.40	3.332×10^{-8}	7.65	1.0049×10^{-14}
0.90	0.18406	3.20	0.00068714	5.45	2.5185×10^{-8}	7.70	6.8033×10^{-15}
0.95	0.17106	3.25	0.00057703	5.50	1.899×10^{-8}	7.75	4.5946×10^{-15}
1.00	0.15866	3.30	0.00048342	5.55	1.4283×10^{-8}	7.80	3.0954×10^{-15}
1.05	0.14686	3.35	0.00040406	5.60	1.0718×10^{-8}	7.85	2.0802×10^{-15}
1.10	0.13567	3.40	0.00033693	5.65	8.0224×10^{-9}	7.90	1.3945×10^{-15}
1.15	0.12507	3.45	0.00028029	5.70	5.9904×10^{-9}	7.95	9.3256×10^{-16}
1.20	0.11507	3.50	0.00023263	5.75	4.4622×10^{-9}	8.00	6.221×10^{-16}

Assuming SD = 1 and mean is 0

http://www.ece.ucdavis.edu/~levy/eec161/qfunc.pdf

Example – Gaussian Distribution

A Gaussian random variable $\mathbf{x}$ has the probability density function

$$f_{\mathbf{x}}(x) = \frac{1}{\sqrt{30\pi}} \exp\left[-(x - 12)^2/30\right]$$

Express the following probabilities in terms of the Q-function:

a. $P(\mathbf{x} \leq 11)$
b. $P(10 < \mathbf{x} \leq 12)$
c. $P(11 < \mathbf{x} \leq 13)$
d. $P(9 < \mathbf{x} \leq 12)$

Example – Gaussian Distribution

A Gaussian random variable x has the probability density function

$$f_x(x) = \frac{1}{\sqrt{30\pi}} \exp[-(x-12)^2/30]$$

Mean = 12 - SD = sqrt (15)

Express the following probabilities in terms of the Q-function:

a. $P(x \le 11)$
b. $P(10 < x \le 12)$
c. $P(11 < x \le 13)$
d. $P(9 < x \le 12)$

$$F_\chi(x)\Big|_{x=11} = \int_{-\infty}^{x=11} N_{m,\sigma}(k)\,dk = 1 - Q\left(\frac{x-m}{\sigma}\right)\Big|_{x=11} = 1 - Q\left(\frac{-1}{\sqrt{30}}\right) = Q\left(\frac{1}{\sqrt{30}}\right)$$

Solution:

a. $P(x \le 11) = Q\left(\frac{12-11}{\sqrt{15}}\right) = Q\left(1/\sqrt{15}\right)$

b. $P(10 < x \le 12) = P\{x \le 12\} - P\{x \le 10\} = Q(0) - Q\left(2/\sqrt{15}\right)$

c. $P(11 < x \le 13) = Q\left(-1/\sqrt{15}\right) - Q\left(1/\sqrt{15}\right) = 1 - 2Q\left(1/\sqrt{15}\right)$

d. $P(9 < x \le 12) = Q(0) - Q\left(3/\sqrt{15}\right) = 0.5 - Q\left(3/\sqrt{15}\right)$

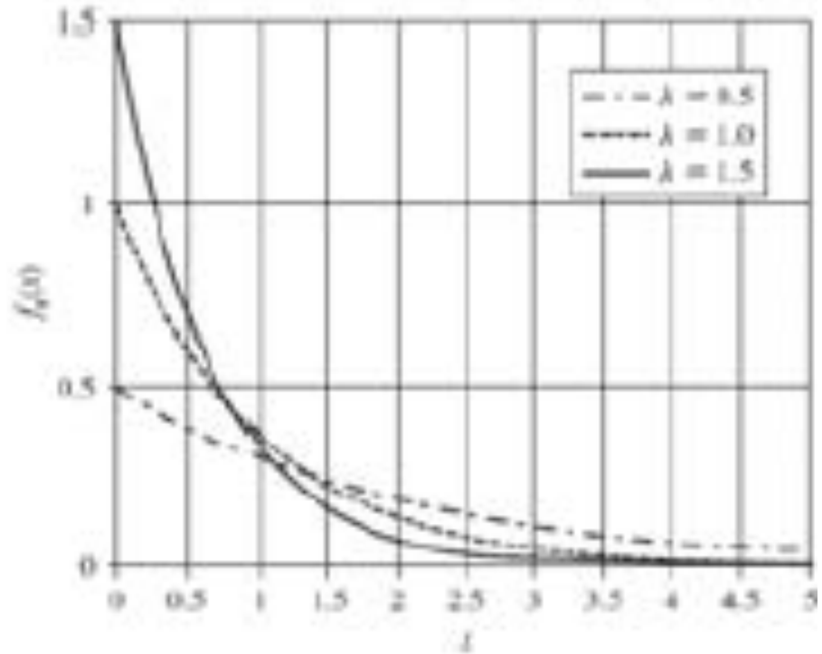
Use table to find the actual values

Exponential Random Variable

- x is an exponential random variable if its PDF is given by

$$f_x(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

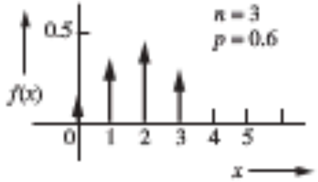
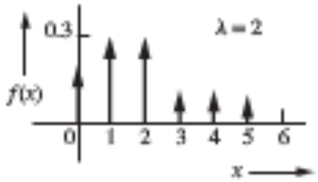
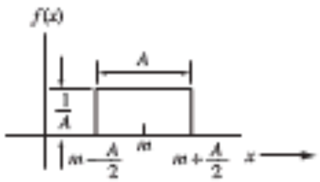
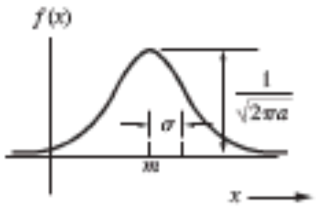
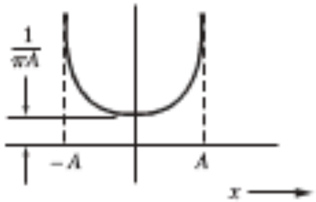
where $\lambda > 0$



- For $x \geq 0$,

$$F_x(x) = P\{x \leq x\} = \int_0^x \lambda e^{-\lambda t} dt = \int_0^x \lambda e^{-\lambda x} dx = -e^{-\lambda t}\Big|_0^x = 1 - e^{-\lambda x}$$

- The exponential random variable is frequently used to model lifetimes (e.g., duration of a phone call) or waiting times (e.g. until some event happens)

Name of Distribution	Type	Sketch of PDF	Equation for: Cumulative Distribution Function (CDF)	Equation for: Probability Density Function (PDF)	Mean	Variance
Binomial	Discrete		$F(a) = \sum_{\substack{k=0 \\ k \le a}}^{n} P(k)$ where $P(k) = \binom{n}{k} p^k (1-p)^{n-k}$	$f(x) = \sum_{k=0}^{n} P(k)\delta(x-k)$ where $P(k) = \binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Poisson	Discrete		$F(a) = \sum_{\substack{k=0 \\ k \le a}}^{n} P(k)$ where $P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$	$f(x) = \sum_{k=0}^{\infty} P(k)\delta(xk)$ where $P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$	λ	λ
Uniform	Continuous		$F(a) = \begin{cases} 0, & a < \left(\frac{2m-A}{2}\right) \\ \frac{1}{A}\left[a - \left(\frac{2m-A}{2}\right)\right], & \lvert a-m \rvert \le \frac{A}{2} \\ 1, & a \ge \left(\frac{2m-A}{2}\right) \end{cases}$	$f(x) = \begin{cases} 0, & x < \left(\frac{2m-A}{2}\right) \\ \frac{1}{A}, & \lvert x-m \rvert \le \frac{A}{2} \\ 0, & x > \left(\frac{2m+A}{2}\right) \end{cases}$	m	$\frac{A^2}{12}$
Gaussian	Continuous		$F(a) = Q\left(\frac{m-a}{\sigma}\right)$ where $Q(\alpha) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-x^2/2}\, dx$	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-(x-m)^2/2\sigma^2]$	m	σ^2
Sinusoidal	Continuous		$F(a) = \begin{cases} 0, & a \le -A \\ \frac{1}{\pi}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{a}{A}\right)\right], & \lvert a \rvert \le A \\ 1, & a \ge A \end{cases}$	$f(x) = \begin{cases} 0, & x < -A \\ \frac{1}{\pi\sqrt{A^2-x^2}}, & \lvert x \rvert \le A \\ 0, & x > A \end{cases}$	0	$\frac{A^2}{2}$

Example

Assume the phase offset between the transmitter and the receiver is modeled by a random variable Theta that is uniformly distributed between [-pi , pi].

a. $P\{\theta \leq 0\}$
b. $P\{\theta \leq \pi/4\}$

This is continuous RV → Find fx(*Theta*)

Example

Assume the phase offset between the transmitter and the receiver is modeled by a random variable Theta that is uniformly distributed between [-pi , pi]. Find

a. $P\{\boldsymbol{\theta} \leq 0\}$
b. $P\{\boldsymbol{\theta} \leq \pi/4\}$

Solution

Because $\boldsymbol{\theta}$ is uniformly distributed between $[-\pi, \pi]$, its PDF is given 0 factorial by

$$f_{\theta}(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \leq \theta \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

a. $$P\{\boldsymbol{\theta} \leq 0\} = \int_{-\infty}^{0} f_{\theta}(\theta)d\theta = \int_{-\pi}^{0} \frac{1}{2\pi}d\theta = \frac{\pi}{2\pi} = \frac{1}{2}$$

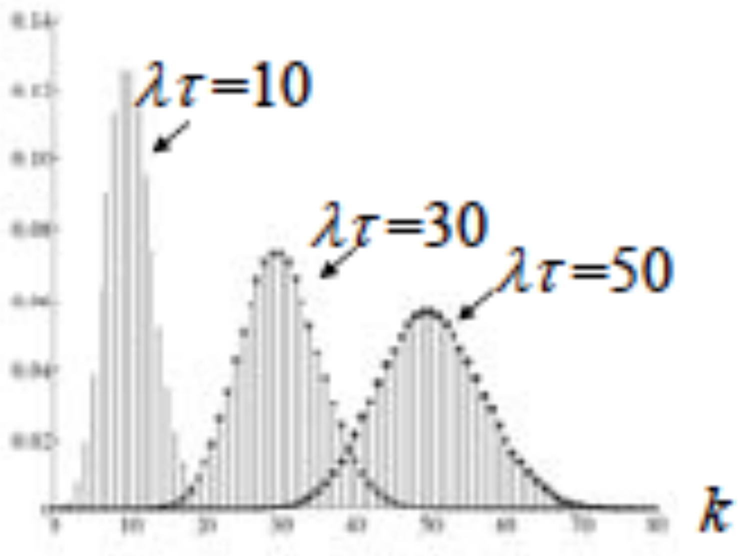
b. $$P\{\boldsymbol{\theta} \leq \pi/4\} = \int_{-\infty}^{\pi/4} f_{\theta}(\theta)d\theta = \int_{-\pi}^{\pi/4} \frac{1}{2\pi}d\theta = \frac{\frac{\pi}{4} + \pi}{2\pi} = \frac{5}{8}$$

Poisson Random Variable

- The Poisson random variable x models the number of events (k) occurring in any interval $(t_o, t_o + \tau)$ if the occurrence of these events, at an average rate λ, is independent of t_o and depends only on the length of interval τ
- It is common in the literature to refer to the occurrence of a Poisson event as an arrival
- x is a Poisson random variable if its PMF is of the form

$$
\begin{aligned}
p_x(k) &= P(x = k) \\
&= P\{k \text{ arrivals in interval } \tau\} \\
&= e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}, \quad k = 0, 1, 2, \cdots, \infty
\end{aligned}
$$

where λ = average arrival rate



Statistics of RV

- Finding behaviors using certain averages
  - Mean, Variance, Standard Deviation, Moments, Central Moments, etc.

- The *expected value* or *mean* of a continuous random variable x is defined as

$$m_x = \overline{x} = E\{x\} = \int_{-\infty}^{+\infty} x\, f_x(x)dx$$

- The expected value of a random variable represents its average value in a very large number of trials
- The mean of the function $y = g(x)$ is

$$\overline{g(x)} = E\{g(x)\} = \int_{-\infty}^{+\infty} g(x)\, f_x(x)dx$$

- The variance $Var(x)$ of a random variable x is defined as

$$Var(x) = \sigma_x^2 = E\{(x - m_x)^2\} = \int_{-\infty}^{+\infty} (x - m_x)^2 f_x(x)dx \geq 0$$

Describes the spread of its PDF around the expected value

Statistics of RV (cont.)

- Variance
- Root-Mean-Square

$$Var(x) = \int_{-\infty}^{\infty} (x^2 - 2xm_x + m_x^2) f_x(x) dx$$

$$= \int_{-\infty}^{\infty} x^2 f_x(x) dx - 2m_x \int_{-\infty}^{\infty} x f_x(x) dx + m_x^2$$

$$= E\{x^2\} - m_x^2 = \overline{x^2} - \bar{x}^2$$

- Note that when mean is zero variance is the same as RMS:

$$Var(x) = E\{x^2\}$$

- Standard Deviation of a RV is

$$\sigma_x = \sqrt{Var(x)}$$

Moments of a RV

- Expected value E{x} is the First Moment of a RV
- RMS value E{x^2} is the Second Moment of a RV
- The n^{th} moment of a real-valued random variable x is

$$E\{x^n\} = \int_{-\infty}^{\infty} x^n f_x(x)dx$$

- The n^{th} central moment of a real-valued random variable x is

$$E\{(x - m_x)^n\} = \int_{-\infty}^{\infty} (x - m_x)^n f_x(x)dx$$

- Hence the variance Var (x) is the second central moment of x

$$Var(x) = E\{x^2\} - m_x^2 = \overline{x^2} - \overline{x}^2$$

Example 1 – Mean & Variance

Find the mean and variance of exponential random variable $\boldsymbol{x}$ with PDF

$$f_{\boldsymbol{x}}(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\lambda > 0$.

Example 1 – Mean & Variance

Find the mean and variance of exponential random variable $\mathbf{x}$ with PDF

$$f_{\mathbf{x}}(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\lambda > 0$.

- The nth moment (integ. by part):

$$E\{\mathbf{x}^n\} = \lambda \int_0^\infty x^n e^{-\lambda x} dx$$

Zero

$$\lambda \int_0^\infty x^n e^{-\lambda x} dx = \lambda \left[-\frac{1}{\lambda} x^n e^{-\lambda x} \Big|_0^\infty + \frac{n}{\lambda} \int_0^\infty x^{n-1} e^{-\lambda x} dx \right]$$

$$E\{\mathbf{x}^n\} = \left[n \int_0^\infty x^{n-1} e^{-\lambda x} dx \right] = \frac{n}{\lambda} E\{\mathbf{x}^{n-1}\}$$

- Thus, for n=0→E{x^0}=1 (*)

For $n = 1$, we have $E\{\mathbf{x}\} = \overline{\mathbf{x}} = \frac{1}{\lambda} E\{\mathbf{x}^0\} = \frac{1}{\lambda}$

For $n = 2$, we have $E\{\mathbf{x}^2\} = \overline{\mathbf{x}^2} = \frac{2}{\lambda} E\{\mathbf{x}\} = \frac{2}{\lambda}\frac{1}{\lambda} = \frac{2}{\lambda^2}$

$Var(\mathbf{x}) = \overline{\mathbf{x}^2} - \overline{\mathbf{x}}^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$ = Second moment – first moment square!

Integration Table (number 57 – Ingration by part)
http://www.sonoma.edu/users/f/farahman/sonoma/courses/es430/resources/integral-table.pdf

(*) $\int e^{ax} dx = \frac{1}{a} e^{ax}$

Paired Random Variables - CDF

- Random experiments where the outcomes are described by a pair of random variables x and y
  - Example: the cumulative GPA (x) and SAT score (y) of a graduating high school senior in CA!
  - Signal x emitted by a radio transmitter and the corresponding signal y that eventually arrives at the receiver

- The **joint cumulative distribution function (CDF)** of two random variables $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$F_{xy}(x, y) = P\{\mathbf{x} \le x,\ \mathbf{y} \le y\}$$

- Note that $F_{xy}(x, y)$ measures the probability of event

$$A = \{\xi \in \Omega : \mathbf{x}(\xi) \le x, \mathbf{y}(\xi) \le y\}$$

Example: $$F_{xy}(0.1, 1.5) = \int_0^{0.1} \int_0^{1.5} f_{xy}(x, y)\,dx\,dy$$

Properties:

(a) $0 \le F_{xy}(x, y) \le 1$

(b) $F_{xy}(\infty, \infty) = 1$

(c) $F_{xy}(x, -\infty) = F_{xy}(-\infty, y) = 0$

(d) $F_{xy}(x, y)$ is nondecreasing

Paired Random Variables - PDF

Joint Probability Density Function

- The **joint probability density function,** $f_{xy}(x, y)$**,** of two random variables ***x*** and ***y*** is defined as

$$f_{xy}(x,y) = \frac{\partial^2 F_{xy}(x,y)}{\partial x\, \partial y}$$

$$\Rightarrow F_{xy}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{xy}(u,v)\, du dv$$

Properties:

(a) $f_{xy}(x,y) \geq 0$ for all (x,y)

(b) $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{xy}(x,y)\, dx dy = F_{xy}(\infty,\infty) = 1$

(c) For a rectangle $\{a < x \leq b, c < y \leq d\}$ in x-y plane,

$$P\{a < x \leq b, c < y \leq d\} = \int_{a}^{b} \int_{c}^{d} f_{xy}(x,y)\, dx dy$$

Paired Random Variables – Conditional PDF

- The conditional PDF of random variable $\boldsymbol{x}$ given $\{\boldsymbol{y} = y\}$, denoted by $f_x(x|y)$, is defined as

$$f_x(x|y) = f_x(x|\boldsymbol{y} = y) = \frac{f_{xy}(x,y)}{f_y(y)}, \quad f_y(y) > 0$$

- Note that for each y with $f_y(y) > 0$, the conditional PDF $f_x(x|y)$ provides a new probabilistic description of the random variable $\boldsymbol{x}$
- Similarly, we can define

$$f_y(y|x) = f_y(y|\boldsymbol{x} = x) = \frac{f_{xy}(x,y)}{f_x(x)}, \quad f_x(x) > 0$$

Note: It is possible to find $f_y(y)$ from $f_{x,y}(x,y)$ over the given range for x:

$$f_y(y) = \int_0^6 f_{xy}(x,y)dx$$

Statistically Independent RV

- Two random variables $\mathbf{x}$ and $\mathbf{y}$ are said to be **statistically independent** if

$$F_{xy}(x,y) = P\{\mathbf{x} \le x, \mathbf{y} \le y\}$$
$$= P\{\mathbf{x} \le x\}P\{\mathbf{y} \le y\} = F_x(x)F_y(y)$$

- Equivalently, for independent random variables

$$f_{xy}(x,y) = f_x(x)f_y(y)$$

The PDF of $\mathbf{x}$ after knowledge of the event $\{\mathbf{y} = y\}$ same as its PDF before the knowledge

- For independent random variables,

$$f_x(x|y) = \frac{f_{xy}(x,y)}{f_y(y)} = \frac{f_x(x)f_y(y)}{f_y(y)} = f_x(x)$$

$$f_y(y|x) = f_y(y)$$

Statistics of Paired RV

- **Expected value** of $x+y$

$$E\{x+y\}=E\{x\}+E\{y\}$$

  - More generally, expectation is a **linear operator**

$$E\left\{\sum_i \alpha_i x_i\right\}=\sum_i \alpha_i E\{x_i\}$$

- **Variance** of $x+y$

$$Var(x+y)=Var(x)+Var(y)+2E\{(x-m_x)(y-m_y)\}$$

- **Covariance** of x and y

$$Cov(x,y)=E\{(x-m_x)(y-m_y)\}$$

$$\Rightarrow Var(x+y)=Var(x)+Var(y)+2Cov(x,y)$$

Correlation and Covariance of Two RVs

. .

- The **correlation** of two random variables *x* and *y* is defined as

$$R_{xy} = E\{xy\}$$

- It is very simple exercise to prove that

$$Cov(x, y) = E\{xy\} - E\{x\}E\{y\} = R_{xy} - m_x m_y$$

- *x* and *y* are called **uncorrelated** random variables if

$$Cov(x, y) = 0$$

$$\Rightarrow E\{xy\} = E\{x\}E\{y\}$$

- The **correlation coefficient** of two random variables *x* and *y* is defined as

$$\rho_{xy} = \frac{Cov(x, y)}{\sigma_x \sigma_y}$$

Corr. Corf is between 0 & 1
If CC = 0 → two RVs are uncorrelated
If CC >= 0 → two RVs are moving in the same direction
If CC < 0 → two RVs are moving in different directions

i.i.d RVs and Central Limit Theorem

Let x_1, x_2, …. be n independent, identically distributed random variables with finite **mean** and **variance**

We consider their scaled sum→

$$z_n = \frac{\sum_{i=1}^{n}(x_i - m)}{\sigma\sqrt{n}} = \frac{s_n - nm}{\sigma\sqrt{n}}$$

- The CDF of z_n converges to a Gaussian CDF as n approaches ∞, independent of the distribution of random variables x_n
- In a nutshell, the central limit theorem, states that the sum of almost any set of independent and randomly generated random variables rapidly converges to the Gaussian distribution λ
- This explains why the Gaussian distribution arises so commonly in practice to reflect the additive effect of multiple random occurrences

Example 2 – Joint PDF

The joint PDF of two random variables is

$$f_{xy}(x, y) = \begin{cases} C(1 + xy), & 0 \le x \le 6,\ \ 0 \le y \le 5 \\ 0, & \text{otherwise} \end{cases}$$

Find the following:

a. The constant C
b. $F_{xy}(0.1, 1.5)$
c. $f_{xy}(x, 3)$
d. $f_x(x|y)$

Example 3 – Statistical Averages

Outline

- Later

References

- Leon W. Couch II, Digital and Analog Communication Systems, 8th edition, Pearson / Prentice, Chapter 6
- "M. F. Mesiya, ”Contemporary Communication Systems”, 1st ed./2012, 978-0-07-. 338036-0, McGraw Hill. Chapter 6