Chapter 2

Signals and Spectra

Outline

- Properties of Signals and Noise
- Fourier Transform and Spectra
- Power Spectral Density and Autocorrelation Function
- Orthogonal Series Representation of Signals and Noise
- Fourier Series
- Linear Systems
- Bandlimited Signals and Noise
- Discrete Fourier Transform

Waveform Properties

- In communications, the received waveform basically comprises two parts:
  - Desired signal or Information
  - Undesired signal or Noise
- Assuming a signal is deterministic and physically realizable (measureable and contains only real part)
- Waveforms belong to many different categories
  - Deterministic or stochastic
  - Analog or digital
  - Power or energy
  - Periodic or non-periodic

Let’s look at various analog waveform characteristics!

Waveform Characteristics (Definitions)

- Time average Operator

$$\langle[\cdot]\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2}[\cdot]dt$$

- Periodic waveform

$$\omega(t) = \omega(t+T_0) \text{ for all } t$$

- Waveform DC (Direct Current) value

If w(t) is periodic with To, *lim*1/T→1/To

$$W_{dc} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2}\omega(t)dt = \text{Average value}$$

where w(t) and W can be *v* or *i*.

- For a physical waveform the DC value over a finite interval t_1 to t_2

$$W_{dc} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\omega(t)dt$$

Note: $W_{dc} = \langle w(t)\rangle = \overline{w(t)}$

See Notes:2a

- Instantaneous power

Work (Joule)/ Time (second) = Power (Watt)

$$p(t) = power = \frac{work}{time} = \frac{work}{charge}\cdot\frac{charge}{time} = v(t).i(t)$$

- Average power $P = \langle p(t)\rangle = \langle v(t)\cdot i(t)\rangle$
- RMS Value $W_{rms} = \sqrt{\langle\omega^2(t)\rangle}$
- Average power for resistive load is

$$P_{av} = \frac{\langle v^2(t)\rangle}{R} = \langle i^2(t)\rangle R = \frac{V_{rms}^{\;2}}{R} = I_{rms}^{\;2} R = V_{rms} I_{rms}$$

- Average normalized power

Pnorm =Pav, when RLoad=1

$$P_{norm} = \langle\omega^2(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\omega^2(t)dt$$

Note : w(t) can be v(t) or i(t)

$$P_{av} = \langle p(t)\rangle = \langle v(t)\cdot i(t)\rangle$$

Note: $\langle w^2(t)\rangle = W_{rms}$

Real Meaning of RMS

RMS for a set of n components

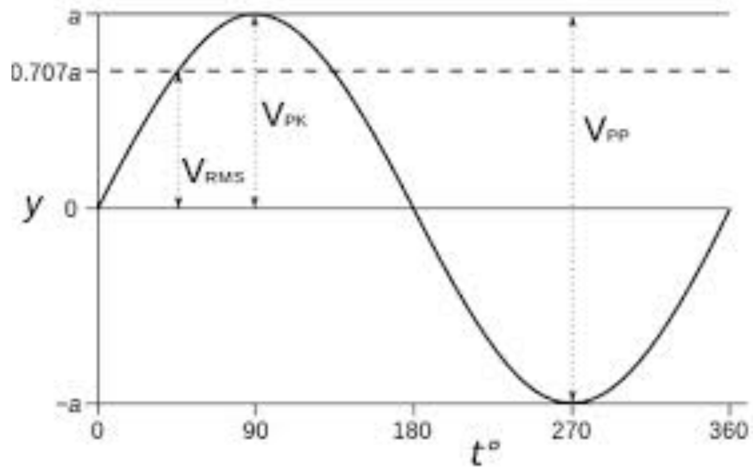
$$x_{\mathrm{rms}} = \sqrt{\frac{1}{n}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)}.$$

RMS for continuous function from T1 to T2

$$f_{\mathrm{rms}} = \sqrt{\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} [f(t)]^2\, dt},$$

RMS for a function over all the times

$$f_{\mathrm{rms}} = \lim_{T\to\infty} \sqrt{\frac{1}{T}\int_0^{T} [f(t)]^2\, dt}.$$



Energy & Power Waveforms

- Average normalized power

$$P = \langle \omega^2(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \omega^2(t)\,dt$$

- Total normalized energy is

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} \omega^2(t)\,dt$$

- w(t) is an energy waveform if & only if total normalized energy is finite & ≠0

Signal Definition:

$$Energy\_Signal \rightarrow 0 < E < \infty$$

$$Power\_Signal \rightarrow 0 < P < \infty$$

Note that a signal can either have Finite total normalized energy or Finite average normalized power

Note:
If w(t) is periodic with To, *lim* 1/T→1/To

Example

- The circuit contains a 120V, 60Hz lamp with in-phase voltage & current waveforms. Find the DC voltage value & the average power.

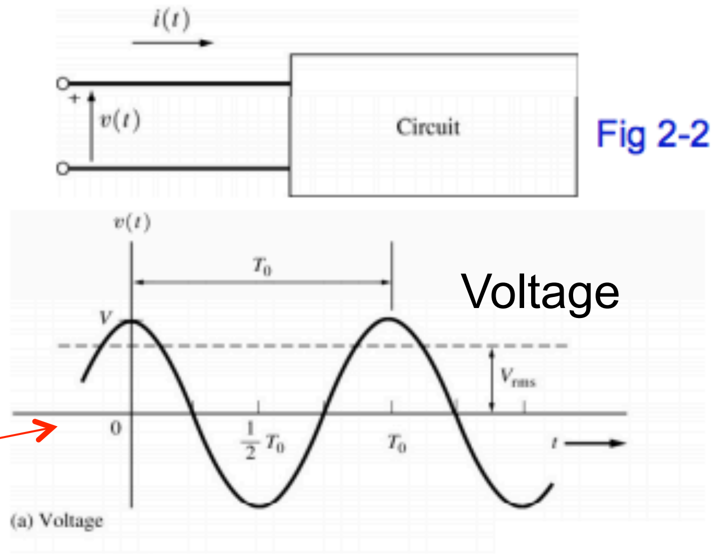


Fig 2-2

Voltage

- DC voltage value:

Periodic Signal!

$$V_{dc} = \langle v(t) \rangle = \langle V\cos(\omega_0 t) \rangle = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} V\cos(\omega_0 t)dt = 0$$

where $\omega_0 = 2\pi / T_0$ & $f_0 = 1/T_0 = 60$ Hz.

- Similarly $I_{dc} = 0$.

Note: $\int \sin ax\,dx = -\frac{1}{a}\cos ax$

Example (continued)

- The circuit contains a 120V, 60Hz lamp with in-phase voltage & current waveforms. Find the DC voltage value & the average power.

* Instantaneous Power:

$$p(t) = (V\cos\omega_0 t)(I\cos\omega_0 t) =$$

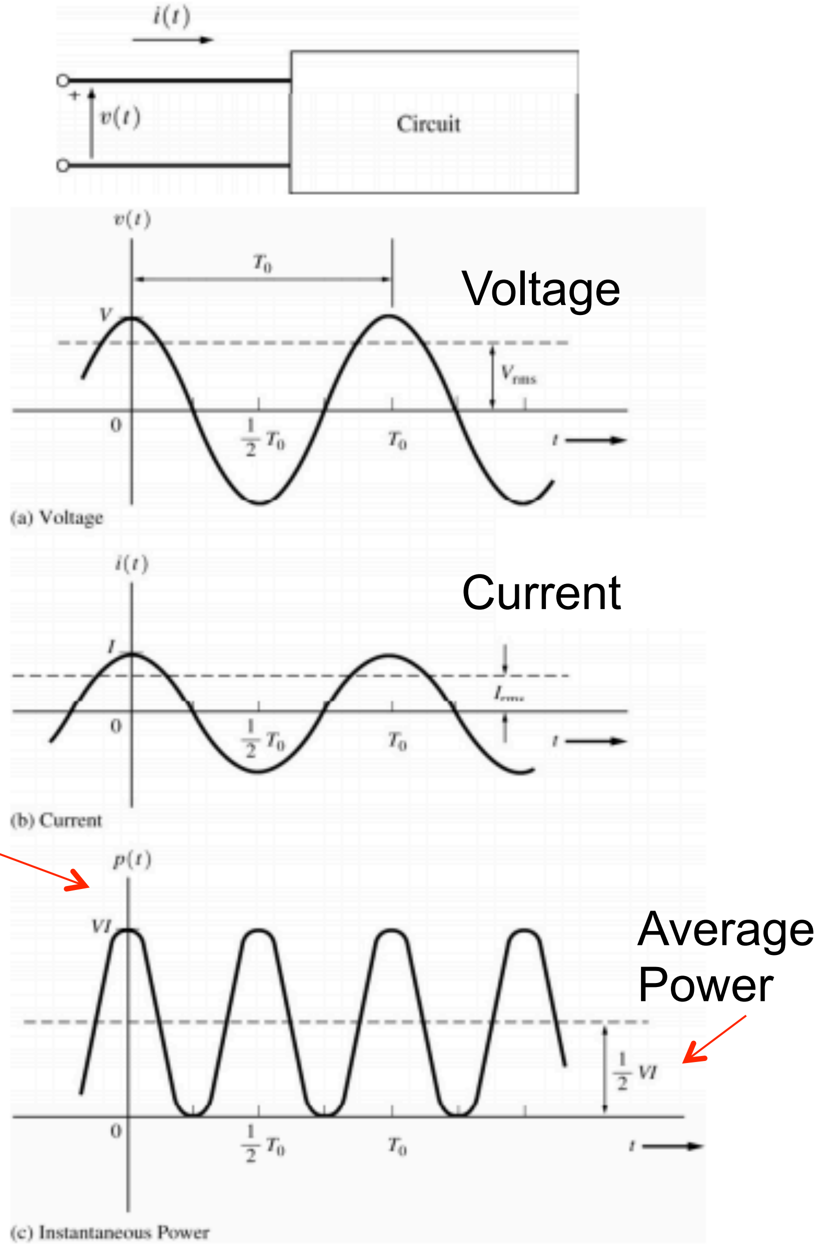
$$\frac{1}{2}VI(1 + \cos 2\omega_0 t)$$

Using Power-Reducing/Half Angle identity

- Average power:

$$P_{ave} = \langle VI\frac{1+\cos 2\omega_0 t}{2}\rangle = \frac{VI}{2T_0}\int_{-T_0/2}^{T_0/2}(1+\cos 2\omega_0 t)dt = \frac{VI}{2}$$

Note: $P_{av} = \langle p(t)\rangle = \langle v(t)\cdot i(t)\rangle$



Example (continued)

- The circuit contains a 120V, 60Hz lamp with in-phase voltage & current waveforms. Find the DC voltage value & the average power.

- RMS values:

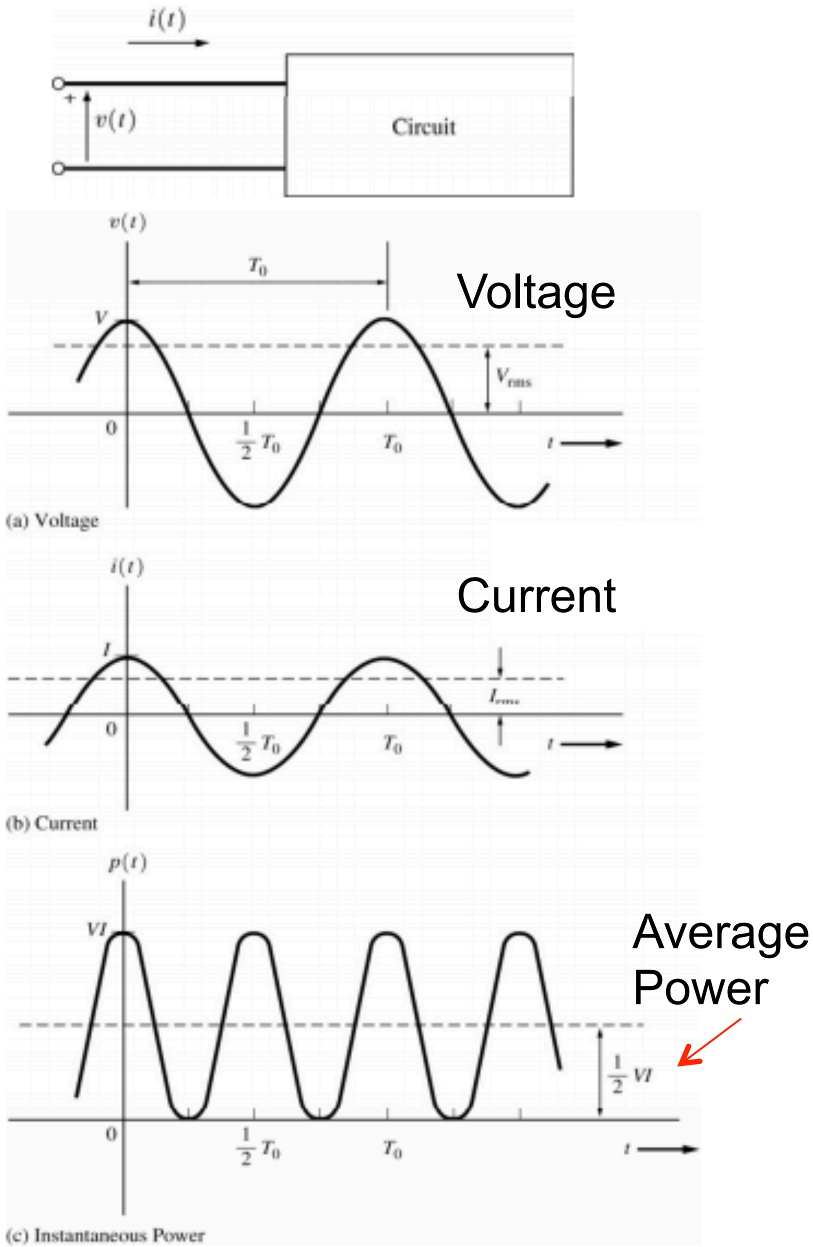
$$V_{rms} = V/\sqrt{2}\ ,\ I_{rms} = I/\sqrt{2}\ ,\ \text{and}\ P_{ave} = \frac{1}{2}VI$$

Note that this is only true when V(t) is a sinusoidal. In this case V is the Peak amplitude of v(t)

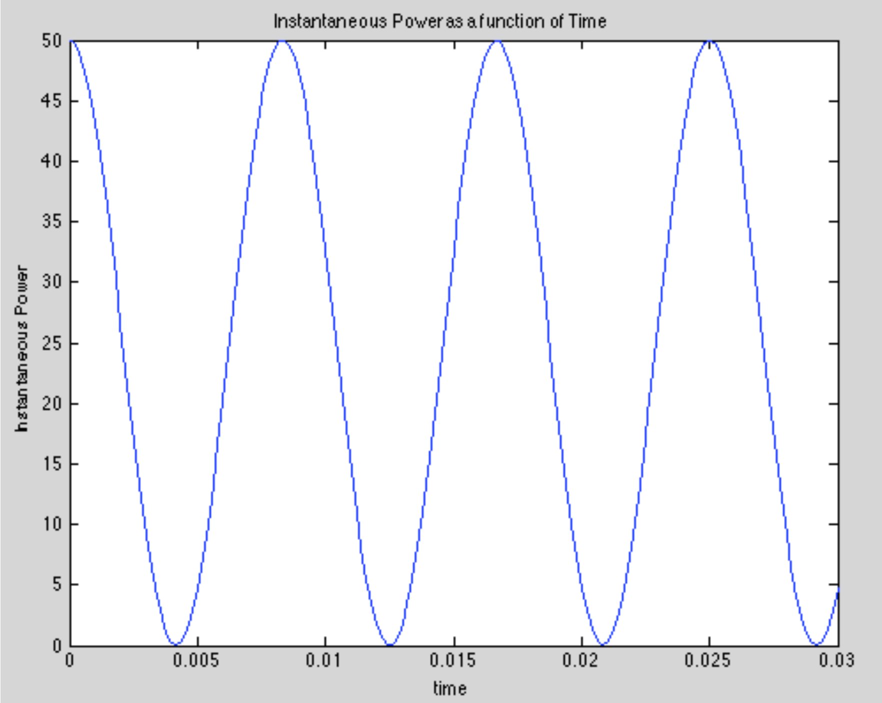
$$V_{rms} = \sqrt{\langle v^2(t)\rangle} = \sqrt{\frac{1}{T_0}\int_{-To/2}^{To/2}[V\cos(w_o t)]^2\,dt}$$

$$V_{rms} = \frac{V}{\sqrt{2}};I_{rms} = \frac{I}{\sqrt{2}};$$ V = Vpeak

$$P_{av} = V_{rms}\cdot I_{rms} = \frac{V\cdot I}{2}$$



Example - Matlab

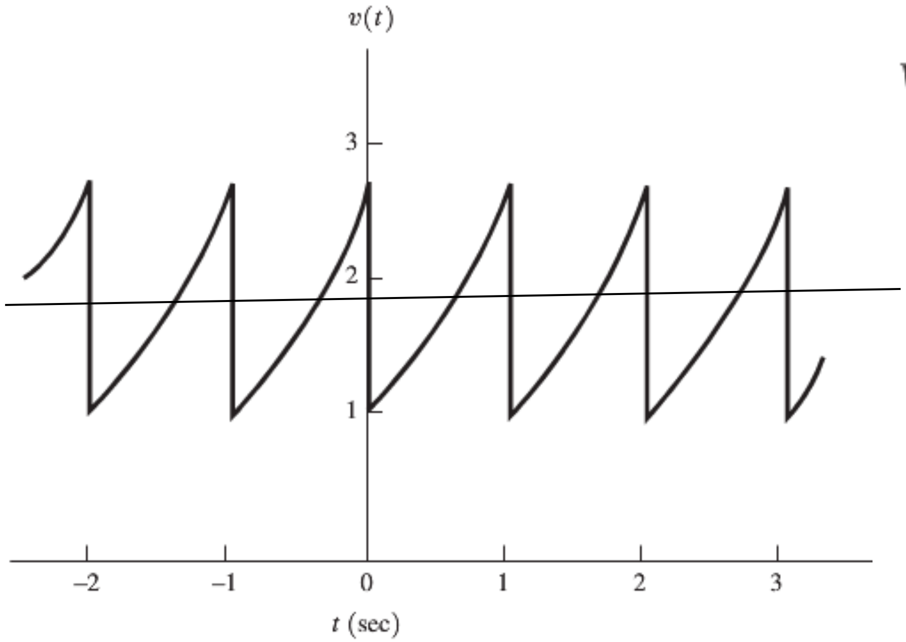


Assume v(t) and i(t) are in phase.
Plot the p(t).

```
clear;

fo = 60;
t = 0:0.0002:0.03;
wo = 2*pi*fo;

% Select theta to be the phase shift of current in degrees
theta = 0;

current = 5*cos(wo*t + theta*(pi/180));
volts = 10*cos(wo*t);
a = 1/60

% Using Eq. (2-6)
instpower = current.*volts;

plot(t,instpower);
xlabel('time');
ylabel('Instantaneous Power');
title('Instantaneous Power as a function of Time');
```

Example

- v(t)= e^t is a periodic voltage signal over time interval 0<t<1. Find DC & RMS values of the waveform



$$V_{dc} = \langle v(t) \rangle = \frac{1}{T_0}\int_0^{T_0} v(t)\,dt = \int_0^1 e^t dt = e^1 - e^0$$

$$V_{dc} = e - 1 = 1.72\,\text{V}$$

$$V_{rms}^2 = \langle v^2(t) \rangle = \int_0^1 (e^t)^2 dt = \frac{1}{2}(e^2 - e^0) = 3.19$$

$$V_{rms} = \sqrt{3.19} = 1.79\,\text{V}$$

Decibel

Decibel is logarithm of power ratio.

$$dB = 10\log_{10}\left(\frac{avePower_{out}}{avePower_{in}}\right) = 10\log_{10}\left(\frac{P_{out}}{P_{in}}\right)$$

For resistive load

$$dB = 20\log_{10}\left(\frac{V_{rms\ out}}{V_{rms\ in}}\right) + 10\log_{10}\left(\frac{R_{in}}{R_{load}}\right)$$

$$dB = 20\log_{10}\left(\frac{I_{rms\ out}}{I_{rms\ in}}\right) + 10\log_{10}\left(\frac{R_{load}}{R_{in}}\right)$$

For normalized powers, $R_{in} = R_{out}$, then

$$dB = 20\log_{10}\left(\frac{V_{rms\ out}}{V_{rms\ in}}\right) = 20\log_{10}\left(\frac{I_{rms\ out}}{I_{rms\ in}}\right)$$

Given dB, the power ratio is $\frac{P_{out}}{P_{in}} = 10^{dB/10}$

The decibel signal-to-noise ratio is

$$(S/N)_{dB} = 10\log_{10}\left(\frac{P_{signal}}{P_{noise}}\right) = 10\log_{10}\left(\frac{\langle s^2(t)\rangle}{\langle n^2(t)\rangle}\right)$$

Because the signal power is

$$\langle s^2(t)\rangle / R = V_{rms\ signal}^2 / R$$

and noise power is

$$\langle n^2(t)\rangle / R = V_{rms\ noise}^2 / R$$

This definition is equivalent to

$$(S/N)_{dB} = 20\log_{10}\left(\frac{V_{rms\ signal}}{V_{rms\ noise}}\right)$$

dBm is decibel power level w.r.t. 1mW:

$$dBm = 10\log_{10}\left(\frac{actualPowerLevel(watts)}{10^{-3}}\right)$$

$$= 30 + 10\log[actualPowerLevel(watts)]$$

One can also define dBmV for voltage:

$$dBmV = 20\log\left(\frac{V_{rms}}{10^{-3}}\right)$$

dBW is decibel power level w.r.t. 1W.

Example

The periodic voltage waveform appears across a *600Ω* resistive load. Find average power dissipated in the load & corresponding dBm value.

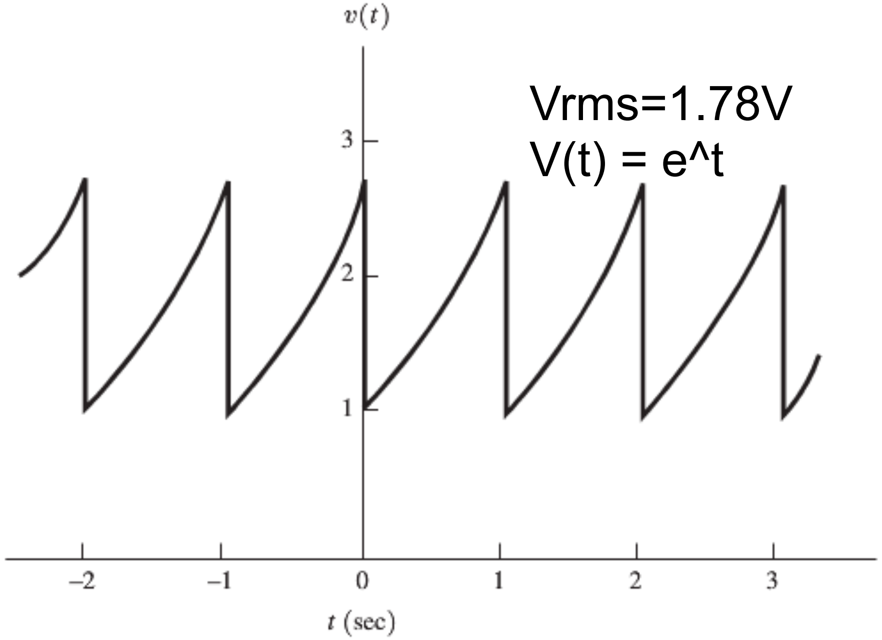
$$P = V_{rms}^2 / R = (1.79)^2 / 600 = 5.32\ mW \text{ and}$$

$$10\log\left(\frac{P}{10^{-3}}\right) = 10\log\left(\frac{5.32x10^{-3}}{10^{-3}}\right) = 7.26dBm$$

Note: The peak instantaneous power is

$$\max[p(t)] = \max[v(t)i(t)] = \max[v(t)^2 / R]$$

$$= (e)^2 / 600 = 12.32mW$$



Vrms=1.78V
V(t) = e^t

Fourier Transform (1)

- How can we represent a waveform?
  - Time domain
  - Frequency domain → rate of occurrences
- Fourier Transform (FT) is a mechanism that can find the frequencies *w(t)*:

$$W(f) = \mathscr{F}[w(t)] = \int_{-\infty}^{\infty} [w(t)]e^{-j2\pi f t} dt$$

- W(f) is the two-sided spectrum of w(t) → positive/neg. freq.
- W(f) is a complex function:

$$W(f) = \underbrace{X(f) + jY(f)}_{\text{Quadrature Components}} = \underbrace{|W(f)|\, e^{j\theta(f)}}_{\text{Phasor Components}} = \sqrt{X^2(f) + Y^2(f)},\ \theta(f) = \tan^{-1}\left(\frac{Y(f)}{X(f)}\right)$$

- Time waveform can be obtained from spectrum using Inverse FT

$$w(t) = \int_{-\infty}^{\infty} W(f)e^{j2\pi f t} df$$

Thus, Fourier Transfer Pair: w(t) ←→ W(f)

Dirac Delta and Unit Step Functions

1. Dirac Delta Function

$$\int_{-\infty}^{\infty} \omega(x)\delta(x)dx = \omega(0)$$

where w(x) is continuous at *x=0*.

- Alternative definitions:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1, \quad \delta(x) = \begin{cases} \infty, x = 0 \\ 0, x \neq 0 \end{cases}$$

- Shifting Property of Delta Function

$$\int_{-\infty}^{\infty} \omega(x)\delta(x - x_0)dx = \omega(x_0)$$

2. Unit step function

$$u(t) = \begin{cases} 1, t > 0 \\ 0, t < 0 \end{cases}$$

Note that

$$\int_{-\infty}^{t} \delta(x)dx = u(t), \text{ thus } \frac{du(t)}{dt} = \delta(t)$$

* dih-**rak**

FT Examples (1)

1. Find FT of *impulse delta signal.*

$$F\{\delta(t)\} = D(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^0 = 1$$

Note that in general:

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$$

In our case, to = 0 and f(to) =1

2. Find FT of a *DC waveform* $\omega(t) = 1$

$$F\{1\} = \int_{-\infty}^{\infty} e^{-j\omega t} dt = \delta(\omega)$$

This can be shown by taking the inverse of delta function.

$$F^{-1}\{\delta(\omega)\} = \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} dt = e^0 = 1, Q.E.D.$$

3. Find the spectrum of an exponential pulse.

$$\omega(t) = \begin{cases} e^{-t}, t > 0 \\ 0, t < 0 \end{cases}$$

$$W(f) = \int_0^{\infty} e^{-t} e^{-j\omega t} dt = \frac{-e^{-(1+j\omega)t}}{(1+j\omega)} \Bigg|_0^{\infty} = \frac{1}{(1+j\omega)}$$

The quadrature components are:

$$X(f) = \frac{1}{1 + (2\pi f)^2} \quad \text{and} \quad Y(f) = \frac{-2\pi f}{1 + (2\pi f)^2}$$

The polar components are:

$$|W(f)| = \sqrt{\frac{1}{1 + (2\pi f)^2}} \quad \text{and} \quad \theta(f) = -\tan^{-1}(2\pi f)$$

See Appendix A of the Textbook!

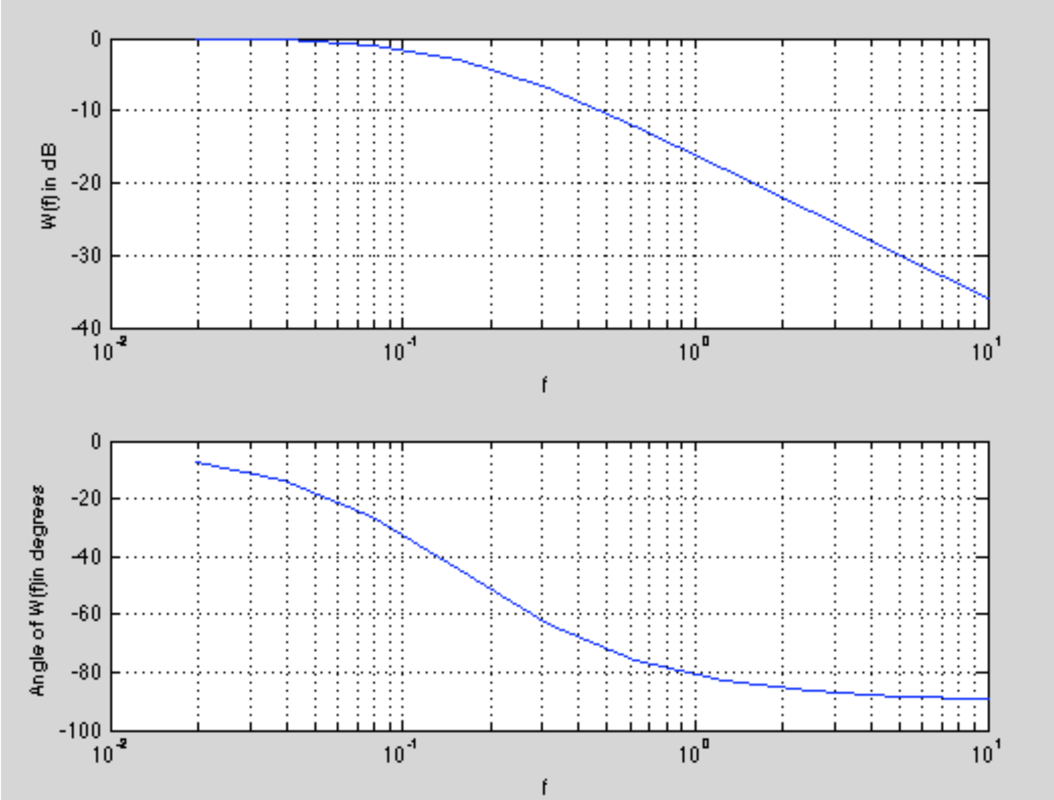
NEXT→

FT Example (2)

```
% The Magnitude-Phase Spectral Functions
% will be plotted.
% The Magnitude function will be plotted in dB units.
% The Phase function will be plotted in degree units.

clear;

for (k = 1:10)
  f(k) = 10*2^(-10)*2^k;
  W(k) = 1/(1 + 2*pi*f(k)*sqrt(-1));
end;

B = log(W);
WdB = (20/log(10))*real(B);
Theta = 180/pi*imag(B);
subplot(211);
semilogx(f,WdB);
xlabel('f');
ylabel('W(f) in dB');
grid;

subplot(212);
semilogx(f,Theta);
xlabel('f');
ylabel('Angle of W(f)in degrees');
grid;
subplot(111);
```

Magnitude-Phase Form:

$$|W(f)| = \sqrt{\frac{1}{1 + (2\pi f)^2}} \quad \text{and} \quad \theta(f) = -\tan^{-1}(2\pi f)$$



Note: Pay attention to how the equations are setup!

Properties of FT

- Spectral symmetry of real signals: If w(t) is real, w(t) = w*(t) then
  - $W(-f) = W^*(f)$, or $|W(f)|$ is even and $\theta(f)$ is odd.
  - $W(f)$ is real when $w(t)$ is even.
  - $W(f)$ is imaginary when $w(t)$ is odd.

- $|W(f)|^2 = \mathscr{E}(f)$ is called *Energy Spectral Density* in Joules/Hz &
- E = integral of $\mathscr{E}(f)$ w.r.t. freq.

- **Parseval's Theorem.**

$$\int_{-\infty}^{\infty} w_1(t) w_2^*(t)\,dt = \int_{-\infty}^{\infty} W_1(f) W_2^*(f)\,df$$

If w1(t)=w2(t)=w(t) →

- **Rayleigh's energy theorem,** which is

$$E = \int_{-\infty}^{\infty} |w(t)|^2 dt = \int_{-\infty}^{\infty} |W(f)|^2 df \rightarrow E = \int_{-\infty}^{\infty} \mathscr{E}(f)\,df$$

Other FT Properties

Operation	Function	Fourier Transform
Linearity	$a_1 w_1(t) + a_2 w_2(t)$	$a_1 W_1(f) + a_2 W_2(f)$
Time delay	$w(t - T_d)$	$W(f) e^{-j\omega T_d}$
Scale change	$w(at)$	$\frac{1}{\lvert a \rvert} W\left(\frac{f}{a}\right)$
Conjugation	$w^*(t)$	$W^*(-f)$
Duality	$W(t)$	$w(-f)$
Real signal frequency translation [$w(t)$ is real]	$w(t)\cos(w_c t + \theta)$	$\frac{1}{2}\left[e^{j\theta} W(f - f_c) + e^{-j\theta} W(f + f_c)\right]$
Complex signal frequency translation	$w(t) e^{j\omega_c t}$	$W(f - f_c)$
Bandpass signal	$\text{Re}\left\{g(t) e^{j\omega_c t}\right\}$	$\frac{1}{2}\left[G(f - f_c) + G^*(-f - f_c)\right]$
Differentiation	$\frac{d^n w(t)}{dt^n}$	$(j2\pi f)^n W(f)$

Find FT of $w(t)\sin(w_c t)$!

$w(t)\sin(w_c t)$ = w(t)(cos(wct-90) = ½ [-j W(f-fc) + j W(f+fc)]

Spectrum of A Sinusoid

- Given v(t) = $A\sin(w_o t)$ the following function plot the magnitude spectrum and phase Spectrum of v(t): |v(f)| & θ(f)

$$v(t) = A\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right)$$

$$V(f) = \int_{-\infty}^{\infty} A\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right) e^{-j\omega t}\,dt$$

$$= \frac{A}{2j}\int_{-\infty}^{\infty} e^{-2j\pi(f-f_0)t}\,dt - \frac{A}{2j}\int_{-\infty}^{\infty} e^{-2j\pi(f+f_0)t}\,dt$$

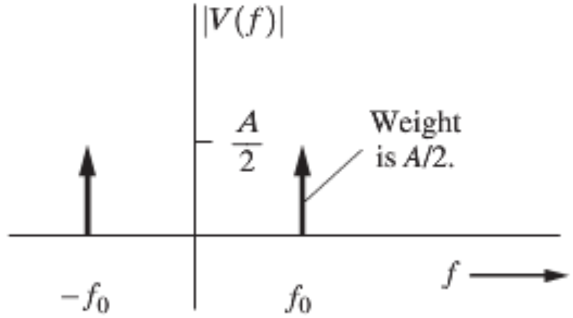
$$= j\frac{A}{2}[\delta(f+f_0) - \delta(f-f_0)]$$

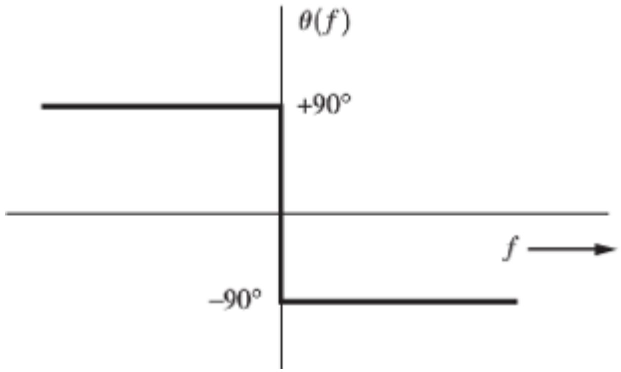
Similar to FT for DC waveform Example

- The magnitude spectrum is

$$|V(f)| = \frac{A}{2}\delta(f-f_0) + \frac{A}{2}\delta(f+f_0)$$

Note that V(f) is purely imaginary
→ When f>0, then θ(f)= -π/2
→ When f<0, then θ(f)= +π/2





Other Fourier Transform Pairs (1)

- **Rectangular pulse:**

$$\Pi\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| \le \frac{T}{2} \\ 0, & |t| > \frac{T}{2} \end{cases}$$

Spectrum of a rectangular pulse

$$W(f) = \int_{-T/2}^{T/2} 1 e^{-j\omega t} dt = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$= T\frac{\sin(\omega T/2)}{(\omega T/2)} = TSa(\pi T f)$$

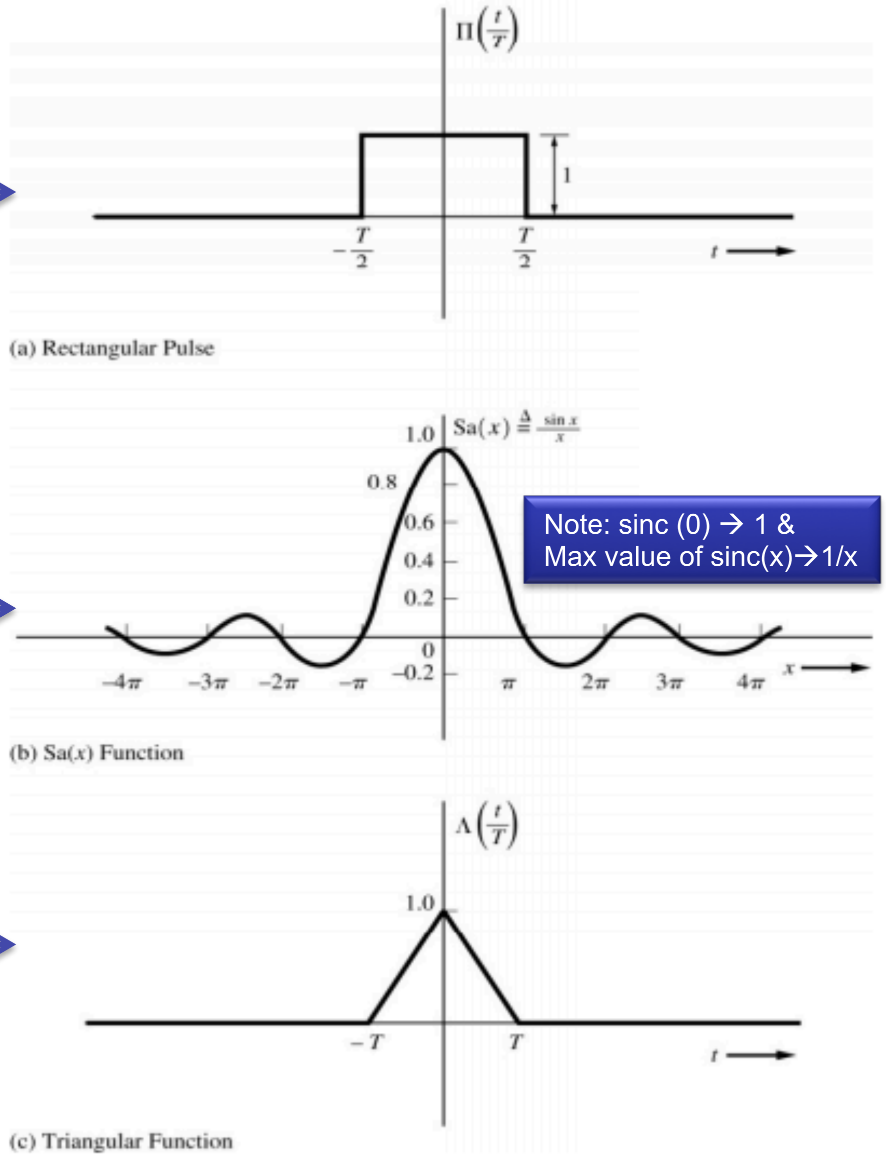
- **Sa (or Sinc) function:**

$$Sa(x) = \frac{\sin(x)}{x} = \text{Sinc}(x/\pi)$$

- **Triangular function:**

$$Tri(t) = \Lambda\left(\frac{1}{T}\right) = \begin{cases} 1 - \frac{|t|}{T}, & |t| \le T \\ 0, & |t| > T \end{cases}$$

$$Tri(t) = \Lambda\left(\frac{t}{T}\right) \leftrightarrow T \cdot Sa^2(\pi f T)$$



(a) Rectangular Pulse

(b) Sa(x) Function

(c) Triangular Function

Sa stands for Sampling Function

Other Fourier Transform Pairs (2)

Function	Time Waveform $w(t)$	Spectrum $W(f)$
Rectangular	$\Pi\left(\frac{t}{T}\right)$	$T[\mathrm{Sa}(\pi f T)]$
Triangular	$\Lambda\left(\frac{t}{T}\right)$	$T[\mathrm{Sa}(\pi f T)]^2$
Unit step	$u(t) \triangleq \begin{cases} +1, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
Signum	$\mathrm{sgn}(t) \triangleq \begin{cases} +1, & t > 0 \\ -1, & t < 0 \end{cases}$	$\frac{1}{j\pi f}$
Constant	1	$\delta(f)$
Impulse at $t = t_0$	$\delta(t - t_0)$	$e^{-j2\pi f t_0}$
Sin c	$\mathrm{Sa}(2\pi W t)$	$\frac{1}{2W}\Pi\left(\frac{f}{2W}\right)$
Phasor	$e^{j(\omega_0 t + \varphi)}$	$e^{j\varphi}\delta(f - f_0)$
Sinusoid	$\cos(\omega_c t + \varphi)$	$\frac{1}{2}e^{j\varphi}\delta(f - f_c) + \frac{1}{2}e^{-j\varphi}\delta(f + f_c)$
Gaussian	$e^{-\pi(t/t_0)^2}$	$t_0 e^{-\pi(f t_0)^2}$
Exponential, one-sided	$\begin{cases} e^{-t/T}, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{2T}{1 + j2\pi f T}$

Examples

1. Using <u>superposition</u>, find the spectrum for a waveform

$$\omega(t) = \Pi\left(\frac{t-5}{10}\right) + 8\sin(6\pi t)$$

<u>Solution</u>: Use rectangular & scaling

$$F\left[\Pi\left(\frac{t-5}{10}\right)\right] = 10\frac{\sin(10\pi f)}{(10\pi f)}e^{-j2\pi f5}$$

Using time delay property

For 8sin(6πt), we have:

Note: 2πfo=2π(3)

$$F[8\sin(6\pi t)] = j\frac{8}{2}[\delta(f+3) - \delta(f-3)]$$

Therefore

$$W(f) = 10\frac{\sin(10\pi f)}{10\pi f}e^{-j10\pi f} + j4[\delta(f+3) - \delta(f-3)]$$

2. Using <u>integration</u>, find the spectrum of

$$\omega(t) = 5 - 5e^{-2t}u(t)$$

<u>Solution:</u>

$$W(f) = \int_{-\infty}^{\infty}\omega(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty}5e^{-j2\pi ft}dt - 5\int_{-\infty}^{\infty}e^{-2t}e^{-j2\pi ft}u(t)dt$$

$$= 5\delta(f) - 5\left.\frac{e^{-2(1+j\pi f)t}}{-2(1+\pi f)}\right|_0^{\infty}, or$$

$$W(f) = 5\delta(f) - \frac{2.5}{1+j\pi f}$$

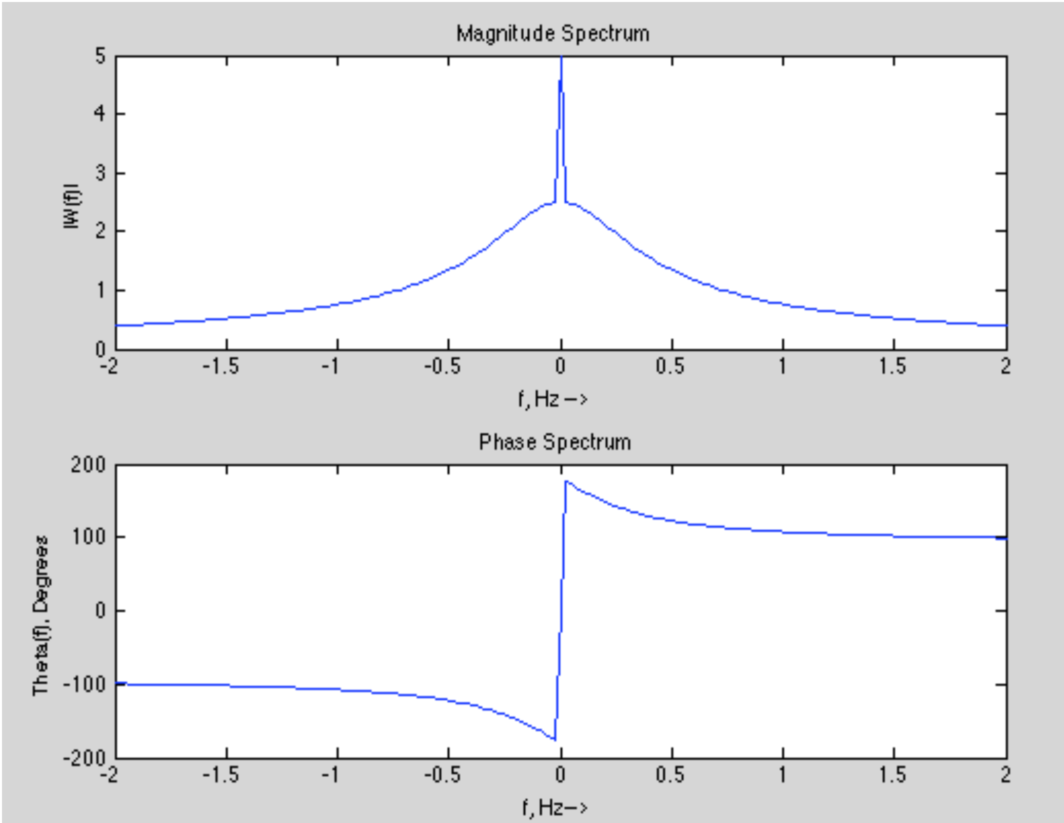
For what freq. W(f) has its max?

Plotting Magnitude and Phase Spectrum

```
% Continuous part of Spectrum
Wmag = zeros(length(f),1);
Theta = zeros(length(f),1);
for (i=1:1:length(f))
   Wmag(i) = abs(-5/(2+2j*pi*f(i)));
   Theta(i)=(180/pi)* angle(-5/(2+2j*pi*f(i)));
end;
```

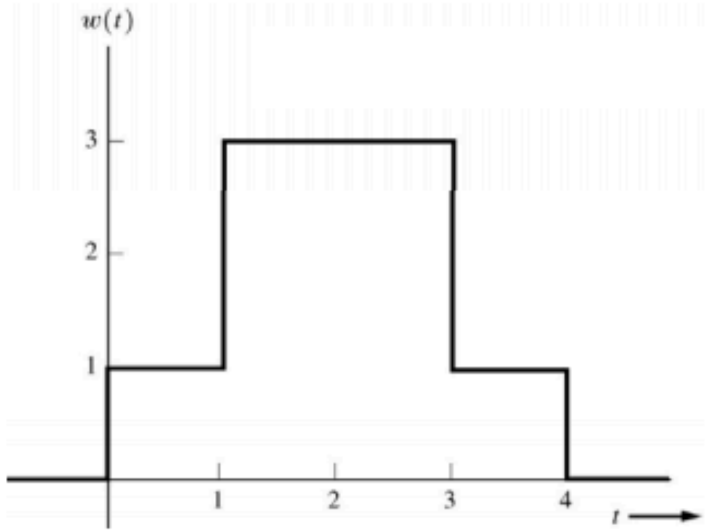
$$W(f) = 5\delta(f) - \frac{2.5}{1 + j\pi f}$$

```
% Modify the Spectrum to include the Delta functions;
for (i = 1:1:length(f))
   if (f(i) == 0)  %only if f(i)=0
    Wmag(i) = 5;
    Theta(i)= 0;
   end;
end;
```



Spectrum of Rectangular Pulses

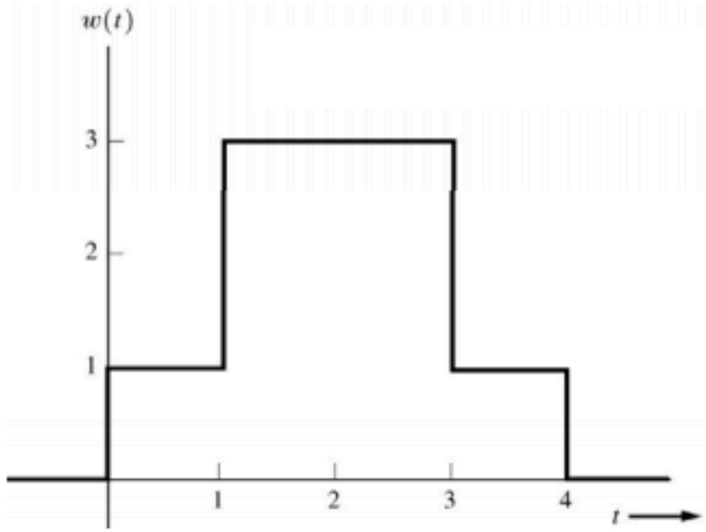
1. Find FT of w(t) waveform



What is w(t)?

Spectrum of Rectangular Pulses

1. Find FT of w(t) waveform



Solution: We use superposition of two rectangular pulses.

$$\omega(t) = \Pi\left(\frac{t-2}{4}\right) + 2\Pi\left(\frac{t-2}{2}\right)$$

From FT tables, we find:

$$W(f) = 4\frac{\sin(4\pi f)}{4\pi f}e^{-j2\omega} + 2(2)\frac{\sin(2\pi f)}{2\pi f}e^{-j2\omega} = 4\left[\mathrm{Sa}(4\pi f) + Sa(2\pi f)\right]e^{-j4\pi f}$$

Power Spectral Density

- How the power content of signals and noise is distributed over different frequencies
- Useful in describing how the power content of signal with noise is affected by filters & other devices
- Important properties:
  - PSD is always a real nonnegative function of frequency
  - PSD is not sensitive to the phase spectrum of w(t) – due to absolute value operation
  - If the PSD is plotted in dB units, the plot of the PSD is identical to the plot of the Magnitude Spectrum in dB units
  - PSD has the unit of watts/Hz (or, equivalently, V^2/Hz or A^2/Hz)

Direct Method!

- PSD for a deterministic power waveform is

$$P_\omega(f) = \left(\lim_{T\to\infty} \frac{|W_T(f)|^2}{T} \right)$$

where $\omega_T(t) \leftrightarrow W_T(f)$ and $P_w(f)$ is in Watts/Hz.

- Normalized average power:

$$P = \langle \omega^2(t) \rangle = \int_{-\infty}^{\infty} P_\omega(f)\, df$$

i.e., the area under PSD function.

Note that $|W(f)|^2$ was the Energy Spectral Density (ESD).

- $W_T(t)$ is the truncated version of the signal:

$$w_T(t) = \begin{cases} w(t), & -T/2 < t < T/2 \\ 0, & t \text{ elsewhere} \end{cases} = w(t)\Pi\left(\frac{t}{T}\right)$$

Any other way we can find PSD?→

Fourier Series

- The complex FS uses the <u>orthogonal</u> exponential function

$$\varphi_n(t) = e^{jn\omega_0 t}$$

where n is any integer, $\omega_0 = 2\pi/T_0$, and $T_0 = (b-a)$ is the length of interval over which the orthogonal series is valid.

- A physical waveform (i.e., finite energy) may be represented over $a < t < a + T_0$

$$\omega(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}, \text{ where}$$

$$c_n = \frac{1}{T_0} \int_a^{a+T_0} \omega(t) e^{-jn\omega_0 t} dt$$

- If $w(t)$ is periodic with period T_0 the series is valid over $-\infty < t < \infty$.

- Properties of FS:
  1. If $w(t)$ is real, $c_n = c_{-n}^*$
  2. If $w(t)$ is real & even, $Im[c_n] = 0$
  3. If $w(t)$ is real & odd, $Re[c_n] = 0$
  4. Pareseval theorem (Avg Pwr)

$$\frac{1}{T_0} \int_a^{a+T_0} |\omega(t)|^2 dt = \sum_{n=-\infty}^{n=\infty} |c_n|^2$$

  5. The complex FS coefficients of a real waveform in quadrature (& polar) from:

$$c_n = \begin{cases} \frac{1}{2}a_n - j\frac{1}{2}b_n = \frac{1}{2}D_n \angle \varphi_n, & n > 0 \\ a_0 = D_0, & n = 0 \\ \frac{1}{2}a_{-n} + j\frac{1}{2}b_{-n} = \frac{1}{2}D_{-n} \angle \varphi_{-n}, & n < 0 \end{cases}$$

FS for Periodic Functions

- We can represent all periodic signals as harmonic series of the form
  - C_n are the Fourier Series Coefficients & n is real
  - n=0 → Cn=o which is the DC signal
  - n=+/-1 yields the fundamental frequency or the first harmonic ω_0
  - |n|>=2 harmonics

$$w(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}$$

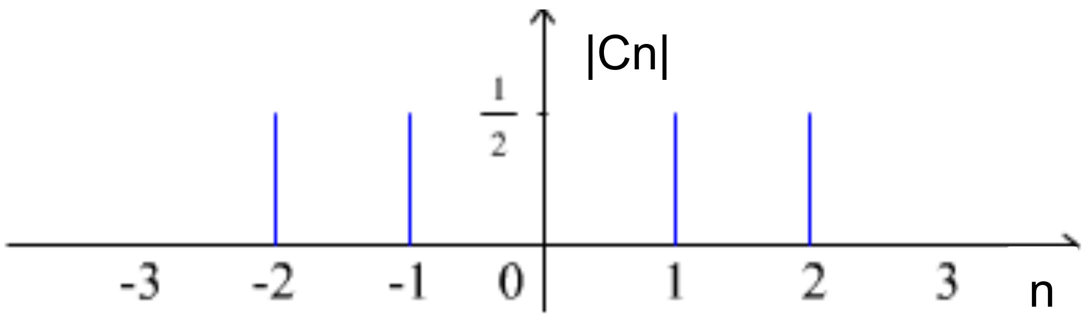
$$c_n = \frac{1}{T_0} \int_a^{a+T_0} w(t) e^{-jn\omega_0 t}\, dt$$

FOR PERIODIC SINUSOIDAL SIGNALS:

$$W(f) = \sum_{n=-\infty}^{n=\infty} c_n\, \delta(f - nf_0)$$

Fourier Series and Frequency Spectra

- We can plot the *frequency spectrum* or *line spectrum* of a signal
  - In Fourier Series n represent harmonics
  - Frequency spectrum is a graph that shows the amplitudes and/or phases of the Fourier Series coefficients *Cn*.
    - Phase spectrum ϕn
    - The lines |Cn| are called line spectra because we indicate the values by lines



Different Forms of Fourier Series

- Fourier Series representation has different forms:

Note that n=k

Polar Form

Quadrature Form

Name	Equation
Exponential	$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}; \quad C_k = \lvert C_k \rvert e^{j\theta_k}, C_{-k} = C_k^*$
Combined trigonometric	$C_0 + \sum_{k=1}^{\infty} 2\lvert C_k \rvert \cos(k\omega_0 t + \theta_k)$
Trigonometric	$A_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$
	$2C_k = A_k - jB_k, C_0 = A_0$
Coefficients	$C_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$

What is the relationship between them? → Finding the coefficients!

Fourier Series in Quadrature & Polar Forms

- In quadrature form over interval $a<t<a+T_0$

$$\omega(t)=\sum_{n=0}^{n=\infty} a_n \cos n\omega_0 t+\sum_{n=1}^{n=\infty} b_n \sin n\omega_0 t, \text{ where}$$

$$a_n=\begin{cases}\dfrac{1}{T_0}\displaystyle\int_a^{a+T_0}\omega(t)dt, & n=0\\ \dfrac{2}{T_0}\displaystyle\int_a^{a+T_0}\omega(t)\cos n\omega_0 t dt, & n\geq 1\end{cases}$$

$$b_n=\frac{2}{T_0}\int_a^{a+T_0}\omega(t)\sin n\omega_0 t dt, \quad n>1$$

Also Known as Trigonometric Form

Slightly different notations!
Note that n=k

- In polar form

$$\omega(t)=D_0+\sum_{n=0}^{n=\infty} D_n\cos(n\omega_0 t+\varphi_n), \text{where}$$

$$D_n=\begin{cases}a_0, & n=0\\ \sqrt{a_n^2+b_n^2}, & n\geq 1\end{cases}=\begin{cases}c_0, & n=0\\ 2|c_n|, & n\geq 1\end{cases}$$

$$\varphi_n=-\tan^{-1}\left(\frac{b_n}{a_n}\right)=\angle c_n, \quad n\geq 1$$

$$a_n=\begin{cases}D_0, & n=0\\ D_n\cos\varphi_n, & n\geq 1\end{cases}$$

$$b_n=-D_n\sin\varphi_n, \quad n\geq 1$$

Also Known as Combined Trigonometric Form

Important Relationships

- Euler's Relationship
  - Review Euler formulas

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$e^{j\theta} = 1\angle\theta$$

$$\arg e^{j\theta} = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right] = \theta$$

Examples of FS

- Find Fourier Series Coefficients for

$$x(t) = \cos(\omega_0 t) + \sin(2\omega_0 t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

$$x(t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2j}e^{j2\omega_0 t} - \frac{1}{2j}e^{-j2\omega_0 t}$$

$$C_1 = \frac{1}{2} \quad C_{-1} = \frac{1}{2} \quad C_2 = \frac{1}{2j} \quad C_{-2} = -\frac{1}{2j}$$

$$C_k = 0, \text{ all other } k.$$

- Find Fourier Series Coefficients for

$$y(t) = \sin^2 2\omega_0 t + 2\cos\omega_0 t = \frac{1}{2}\left(1 - \cos 4\omega_0 t\right) + 2\cos\omega_0 t$$

$$y(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

$$y(t) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}e^{j4\omega_0 t} + \frac{1}{2}e^{-j4\omega_0 t}\right) + 2\left(\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right)$$

$$y(t) = \frac{1}{2} - \frac{1}{4}e^{j4\omega_0 t} - \frac{1}{4}e^{-j4\omega_0 t} + e^{j\omega_0 t} + e^{-j\omega_0 t}$$

$$C_0 = \frac{1}{2} \quad C_4 = -\frac{1}{4} \quad C_{-4} = -\frac{1}{4} \quad C_1 = 1 \quad C_{-1} = 1$$

$$C_k = 0, \text{ all other } k.$$

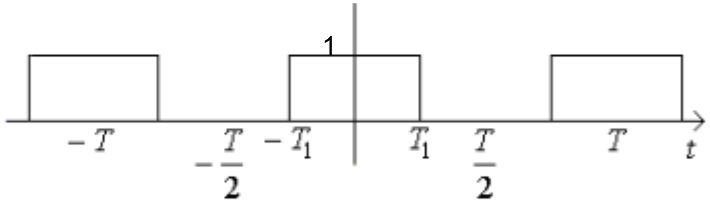
Remember:

1. $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$
2. $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$
3. $\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$
4. $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
5. $\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$
6. $\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$
7. $\sin 2a = 2 \sin a \cos a$
8. $\cos^2 a = \frac{1}{2}(1 + \cos 2a)$
9. $\sin^2 a = \frac{1}{2}(1 - \cos 2a)$

Example

- Given the following periodic square wave, find the Fourier Series representations and plot Ck as a function of k.

(Rectangular waveform)



$$C_k = \frac{1}{T}\int_{-T_1}^{T_1}(1)\,e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1}e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\left(\frac{-1}{jk\omega_0}\right)e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$= \frac{-1}{jk\omega_0 T}\left(e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right)$$

$$= \frac{2}{k\omega_0 T} * \frac{1}{2j}\left(e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right)$$

$$= \frac{2}{k\omega_0 T}\sin(k\omega_0 T_1)$$

$$= \frac{2T_1}{T}\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}$$

Sinc Function

$$= \frac{2T_1}{T}\text{sinc}(k\omega_0 T_1)$$

Note that:
$T_1 = T/4 = T_o/4$
$w_o = 2\pi/T = 2\pi/T_o$

$$C_k = \frac{T}{T_o}\sin c\frac{Tk\omega_o}{2} = \frac{2T_1}{T_o}\sin c(T_1 k\omega_o)$$

$$\rightarrow x(t) = \sum_{k=-\infty}^{\infty}\frac{2T_1}{T_o}\sin c(T_1 k\omega_o)e^{j\omega_o tk}$$

$$X(f) = \sum_{n=-\infty}^{n=\infty} c_n\,\delta(f - nf_0)$$

Note: sinc (infinity) → 1 &
Max value of sinc(x)→1/x

Example

Find the Fourier coefficients for the periodic rectangular wave shown here:

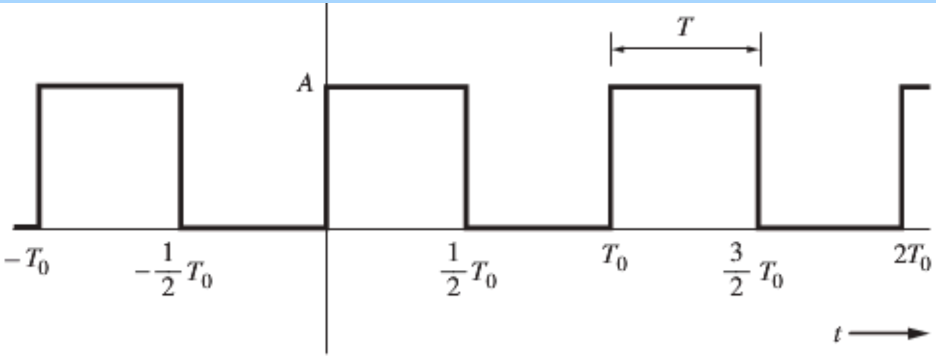
$$c_n = \frac{1}{T_0}\int_0^{T_0/2} Ae^{-jn\omega_0 t}\,dt = j\frac{A}{2\pi n}(e^{-jn\pi} - 1)$$

$$c_n = \begin{cases} \frac{A}{2}, & n = 0 \\ -j\frac{A}{n\pi}, & n = \text{odd} \\ 0, & n \text{ otherwise} \end{cases}$$

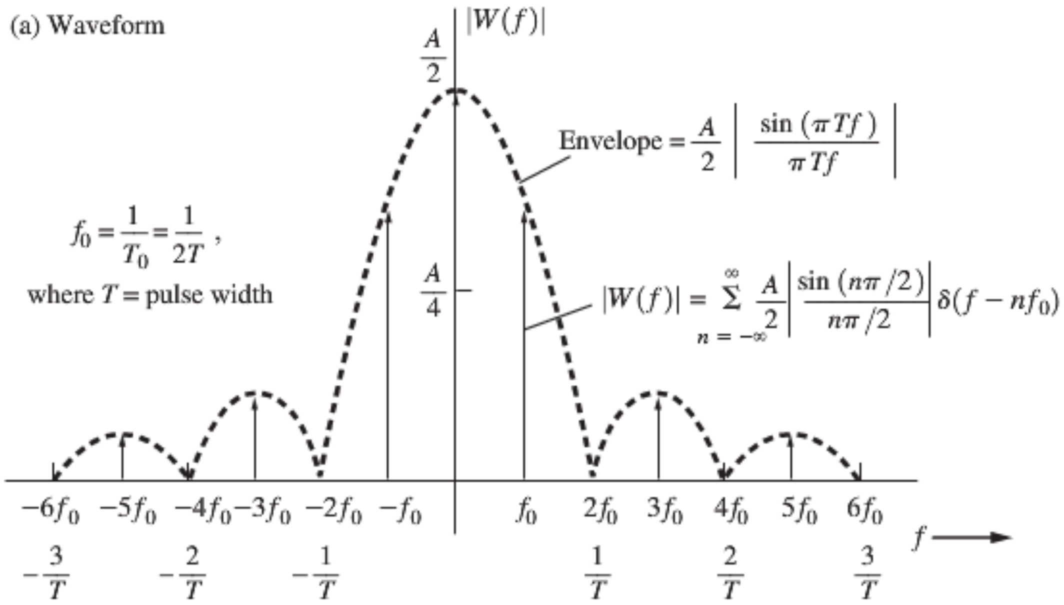
$$C_n = \frac{A}{2}e^{-jn\pi/2}\frac{\sin(n\pi/2)}{n\pi/2}$$

$$W(f) = \sum_{n=-\infty}^{n=\infty} c_n\,\delta(f - nf_0)$$

$$|W(f)| = \sum_{n=-\infty}^{\infty} \frac{A}{2}\left|\frac{\sin(n\pi/2)}{n\pi/2}\right|\delta(f - nf_0)$$



(a) Waveform



Magnitude Spectrum

PSD Of A Periodic Square Waveform

- For a periodic waveform, PSD is

$$PSD = P(f) = \sum_{n=-\infty}^{n=\infty} |c_n|^2 \, \delta(f - nf_0)$$

Therefore, $PSD = \sum_{-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$

$$PSD = P(f) = F[R(\tau)] = F\left[\sum_{-\infty}^{\infty} |c_n|^2 e^{jn\omega_0 \tau}\right]$$

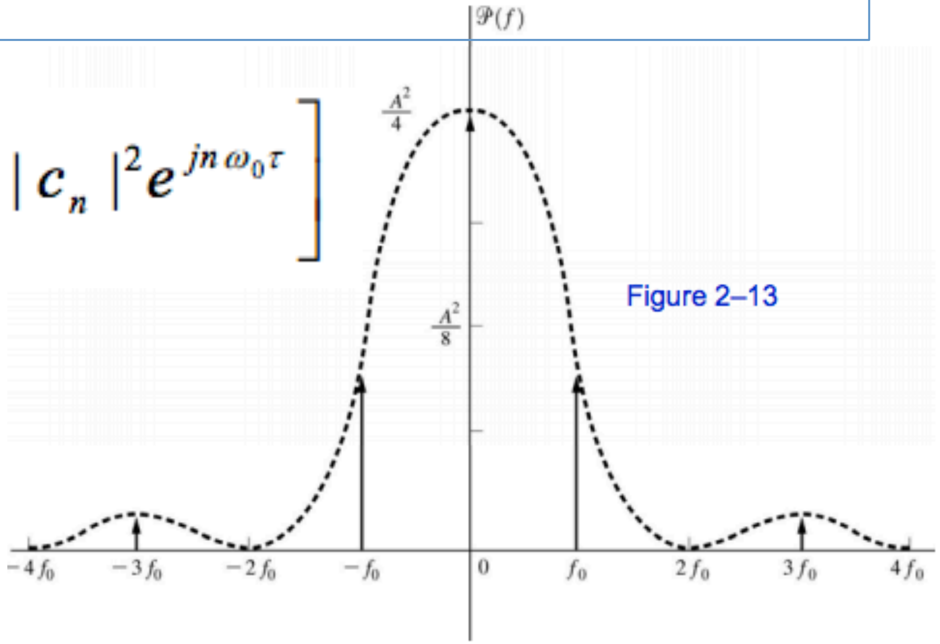


Figure 2–13

Make sure you know the difference between Frequency Spectrum, Magnitude Frequency Spectrum, and Power Spectral Density

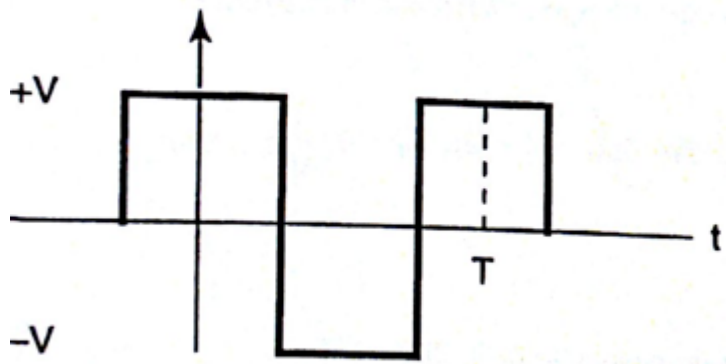
- Example: What is PSD for a square wave?

$$P(f) = \sum_{n=-\infty}^{n=\infty} \left(\frac{A}{2}\right)^2 \left(\frac{\sin(n\pi/2)}{n\pi/2}\right)^2 \delta(f - nf_0)$$

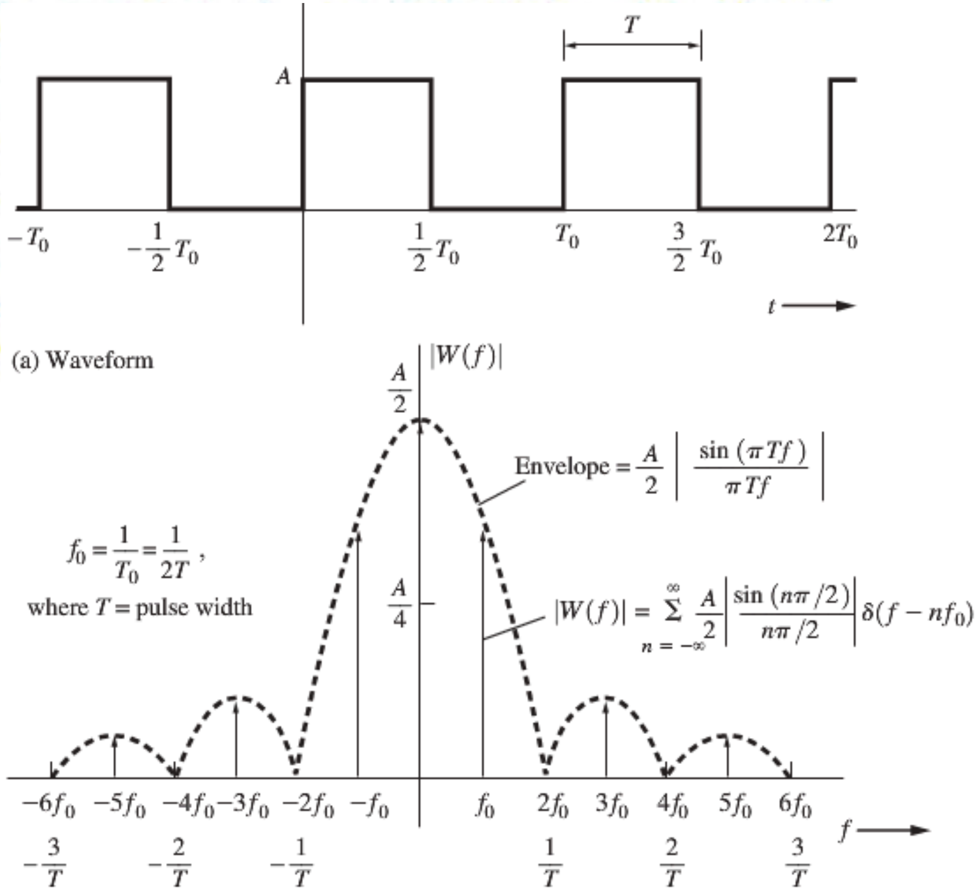
Same Example – A different Approach

- Note that here we are using quadrature form of shifted version of v(t):

$$\omega(t) = \sum_{n=0}^{n=\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{n=\infty} b_n \sin n\omega_0 t,$$



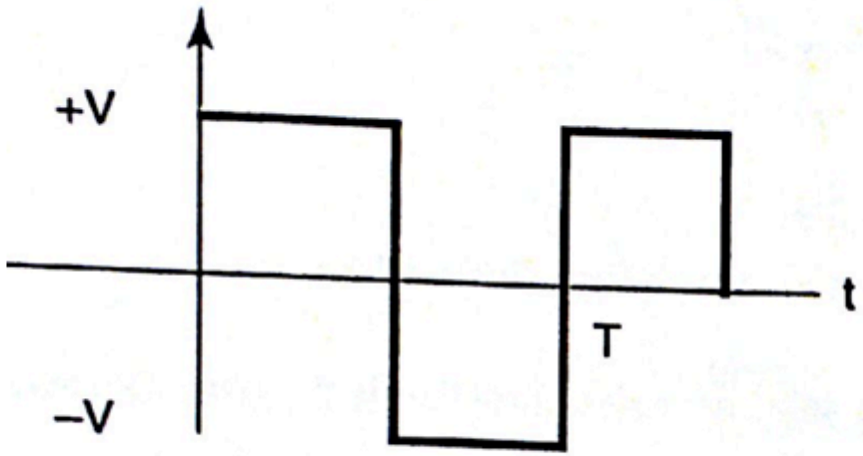
$$v(t) = \sum_{N=\text{odd}}^{\infty} \frac{V \sin N\pi/2}{N\pi/2} \cos N\omega t$$



What is the difference?
Note that N=n; T=To

A Closer Look at the Quadrature Form of FS

- Consider the following quadrature FS representation of an odd square waveform with no offset:

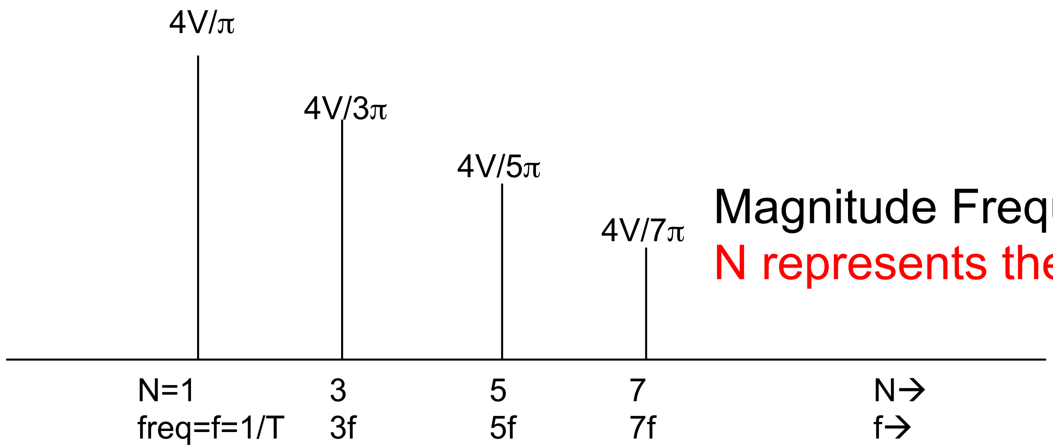


$$v(t) = \frac{4V}{\pi}\sin \omega t + \frac{4V}{3\pi}\sin 3\omega t + \cdots$$

$$v(t) = \sum_{N=\text{odd}}^{\infty} \frac{4V}{N\pi}\sin N\omega t$$

$$W(f) = \sum_{n=-\infty}^{n=\infty} c_n\, \delta(f - nf_0)$$

Thus: Cn = 4V/Nπ

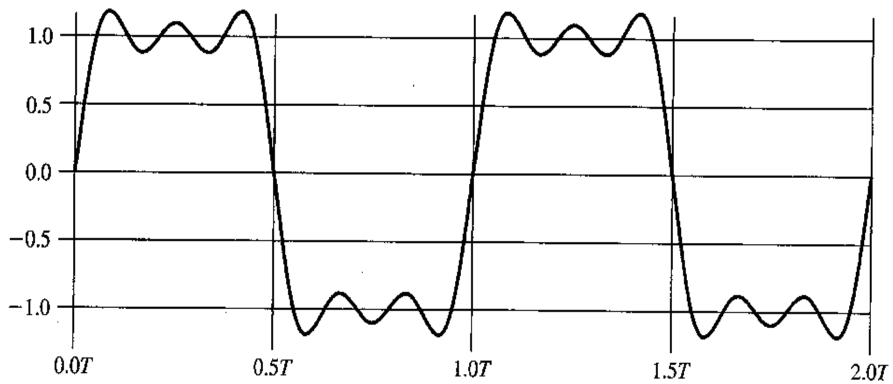


Magnitude Frequency Spectrum W(f)=FS[v(t)]

N represents the Harmonic NUMBER

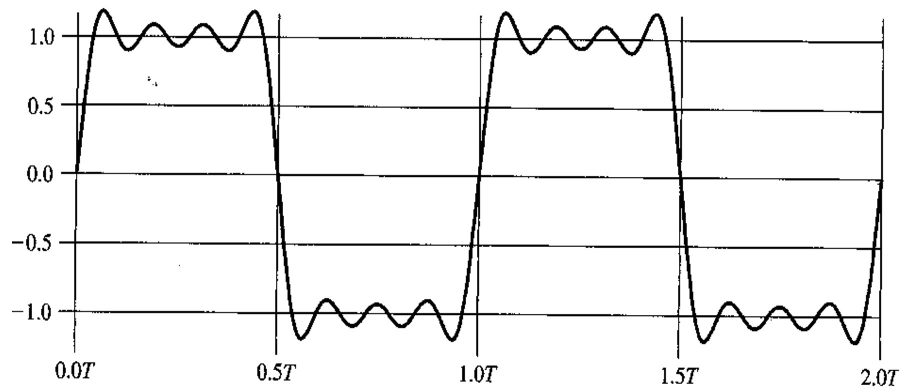
How does the time-domain waveform of the first 7 harmonics look like?

Generating an Square Wave



(a) $(4/\pi)\ [\sin(2\pi f t) + (1/3)\sin(2\pi(3f)t) + (1/5)\sin(2\pi(5f)t)]$

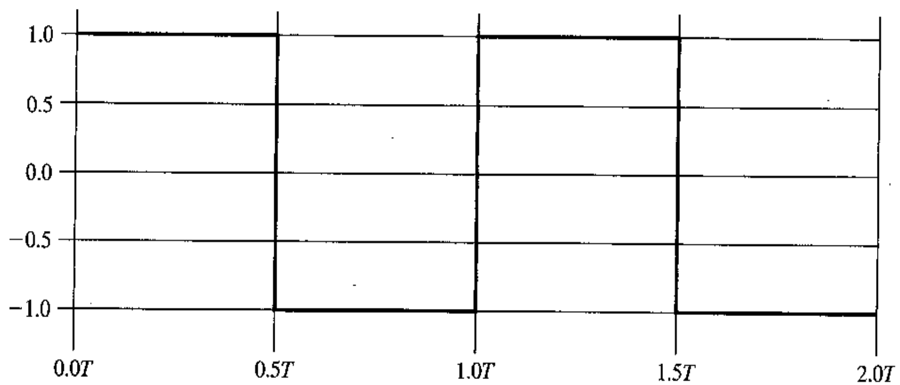
N=1,3,5



(b) $(4/\pi)\ [\sin(2\pi f t) + (1/3)\sin(2\pi(3f)t) + (1/5)\sin(2\pi(5f)t) + (1/7)\sin(2\pi(7f)t)]$

N=1,3,5, 7

This is how the time-domain waveform of the first 7 harmonics looks like!



(c) $(4/\pi)\ \Sigma\ (1/k)\sin(2\pi(kf)t)$, for k odd

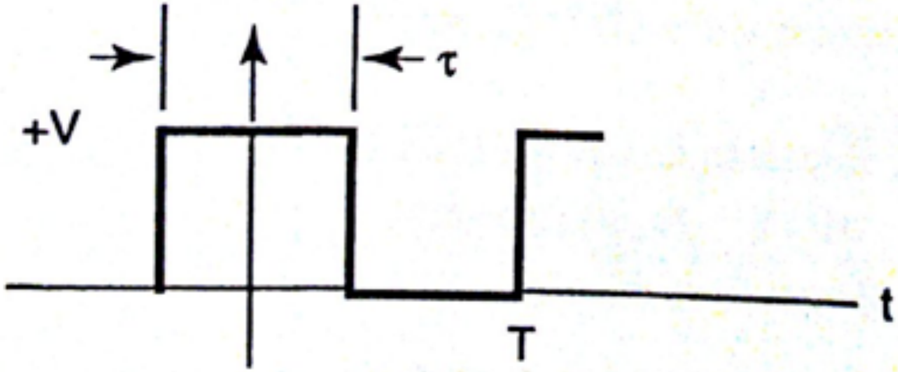
N=1,3,5, 7, 9, …..

Frequency Components of Square Wave

$$v(t) = \sum_{N=\text{odd}}^{\infty} \frac{4V}{N\pi} \sin N\omega t$$

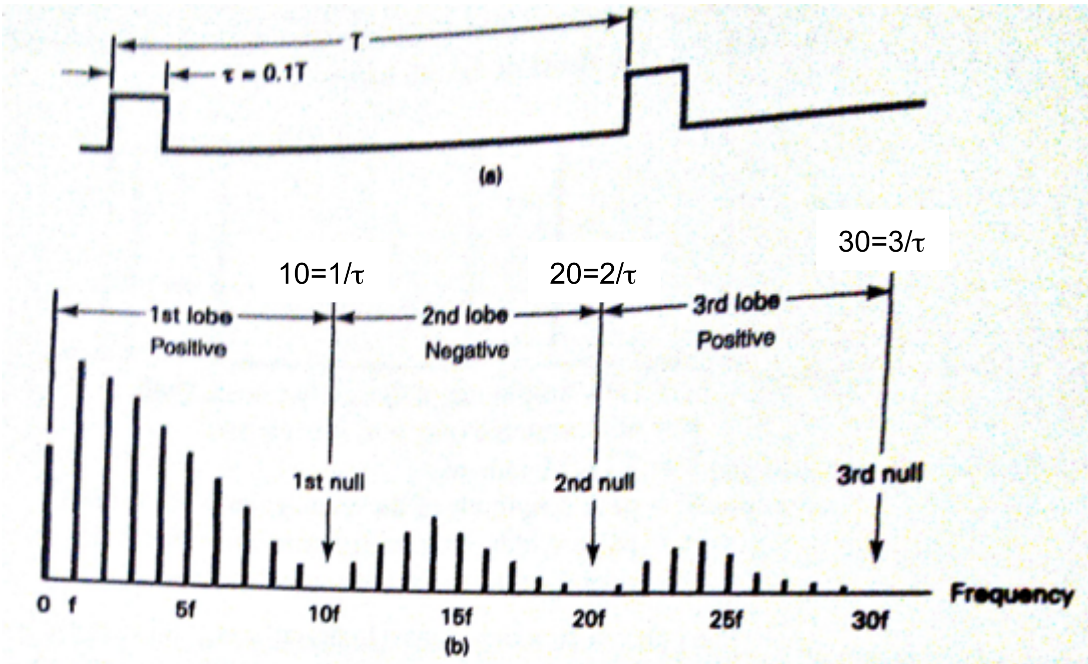
Fourier Expansion

What Is the FS of A Pulse Signal?

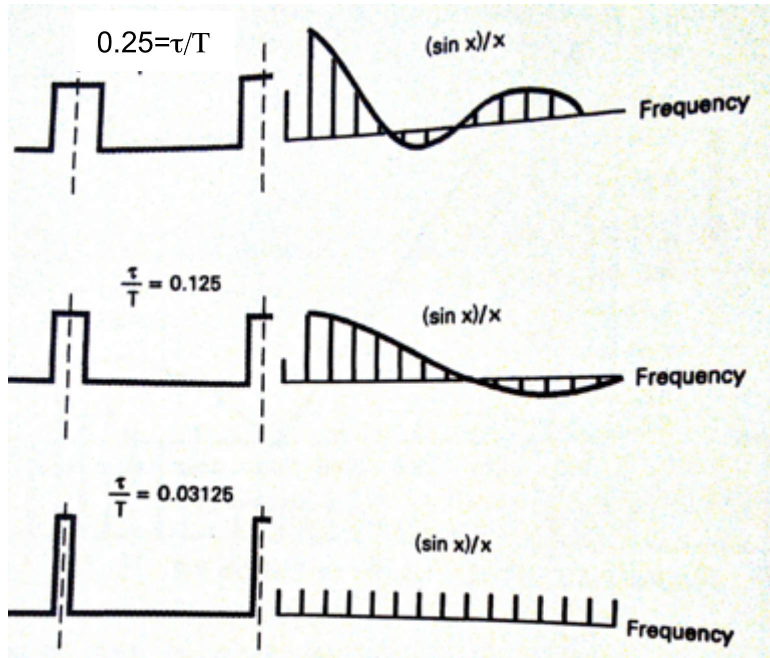


$$v(t) = \frac{V\tau}{T} + \sum_{N=1}^{\infty}\left(\frac{2V\tau}{T}\,\frac{\sin N\omega t/T}{N\pi t/T}\right)\cos N\pi t$$

Note that the width of the pulse can change!



Magnitude Line Spectra of the pulse signal – note that the envelope is a sinc ( ) function!



What happens to the envelope as the pulse gets smaller?

References

- Leon W. Couch II, Digital and Analog Communication Systems, 8th edition, Pearson / Prentice, Chapter 1
- Electronic Communications System: Fundamentals Through Advanced, Fifth Edition by Wayne Tomasi – Chapter 2
  (https://www.goodreads.com/book/show/209442.Electronic_Communications_System)